Elements of Aomoto's generalized hypergeometric functions and a novel perspective on Gauss' hypergeometric differential equation

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Abstract

We review Aomoto's generalized hypergeometric functions on Grassmannian spaces $Gr(k+1, n+1)$. Particularly, we clarify integral representations of the generalized hypergeometric functions in terms of twisted homology and cohomology. With an example of the $Gr(2,4)$ case, we consider in detail Gauss' original hypergeometric functions in Aomoto's framework. This leads us to present a new systematic description of Gauss' hypergeometric differential equation in a form of a first order Fuchsian differential equation.

Contents

1 Introduction

Studies of physical phenomena such as scattering amplitudes (see, *e.g.*, [1]-[5]) and quantum Hall effects [6] in Grassmannian spaces $Gr(k+1, n+1)$ have been attentively carried out in recent years. This has revived an interest in a purely mathematical subject, *i.e.*, generalized hypergeometric functions on Grassmannian spaces, which were introduced and developed by Gelfand [7] and independently by Aomoto [8] many years ago. One of the main goals of this chapter is to present a clear and systematic review on these particular topics in mathematical physics. Particularly, we clarify integral representations of Aomoto's generalized hypergeometric functions in terms of twisted homology and cohomology. As the simplest example, we consider in detail Gauss' original hypergeometric functions in Aomoto's framework so as to familiarize ourselves to the concept of twisted homology and cohomology.

This chapter is organized as follows. In the next section we review some formal results of Aomoto's generalized hypergeometric functions on $Gr(k+1, n+1)$, based on Japanese textbooks [8, 9]. We present a review in a pedagogical fashion since these results are not familiar enough to many physicists and mathematicians. In section 3 we consider a particular case $Gr(2, n+1)$ and present its general formulation. In section 4 we further study the case of $Gr(2, 4)$ which reduces to Gauss' hypergeometric function. Imposing permutation invariance among branch points, we obtain new realizations of the hypergeometric differential equation in a form of a first order Fuchsian differential equation. This is an original result first reported in [10] and this chapter is to be based mainly on some part of [10].

2 Elements of Aomoto's generalized hypergeometric functions

2.1 Definition

Let Z be a $(k+1)\times(n+1)$ matrix

$$Z = \begin{pmatrix} z_{00} & z_{01} & z_{02} & \cdots & z_{0n} \\ z_{10} & z_{11} & z_{12} & \cdots & z_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{k0} & z_{k1} & z_{k2} & \cdots & z_{kn} \end{pmatrix} \tag{2.1}$$

where $k < n$ and the matrix elements are complex, $z_{ij} \in \mathbb{C}$ $(0 \le i \le k\,;\ 0 \le j \le n)$. A function of Z, which we denote $F(Z)$, is defined as *a generalized hypergeometric function on Grassmannian space* $Gr(k+1, n+1)$ when it satisfies the following relations:

$$\sum_{j=0}^{n} z_{ij} \frac{\partial F}{\partial z_{pj}} = -\delta_{ip} F \quad (0 \le i, p \le k) \tag{2.2}$$

$$\sum_{i=0}^{k} z_{ij} \frac{\partial F}{\partial z_{ij}} = \alpha_j F \quad (0 \le j \le n) \tag{2.3}$$

$$\frac{\partial^2 F}{\partial z_{ip} \partial z_{jq}} = \frac{\partial^2 F}{\partial z_{iq} \partial z_{jp}} \quad (0 \le i, j \le k\,;\ 0 \le p, q \le n) \tag{2.4}$$

where the parameters α_j obey the non-integer conditions

$$\alpha_j \notin \mathbb{Z} \quad (0 \le j \le n) \tag{2.5}$$

$$\sum_{j=0}^{n} \alpha_j = -(k+1) \tag{2.6}$$

2.2 Integral representation of $F(Z)$ and twisted cohomology

The essence of Aomoto's generalized hypergeometric function [8] is that, by use of the so-called twisted de Rham cohomology,[1] $F(Z)$ can be written in a form of integral:

$$F(Z) = \int_{\Delta} \Phi \omega \tag{2.7}$$

[1] The *twisted* de Rham cohomology is a version of the ordinary de Rham cohomology into which multivalued functions, such as Φ in (2.8), are incorporated. For mathematical rigor on this, see Section 2 in [8].

where

$$\Phi = \prod_{j=0}^{n} l_j(\tau)^{\alpha_j} \tag{2.8}$$

$$l_j(\tau) = \tau_0 z_{0j} + \tau_1 z_{1j} + \cdots + \tau_k z_{kj} \quad (0 \le j \le n) \tag{2.9}$$

$$\omega = \sum_{i=0}^{k} (-1)^i \tau_i d\tau_0 \wedge d\tau_1 \wedge \cdots \wedge d\tau_{i-1} \wedge d\tau_{i+1} \wedge \cdots \wedge d\tau_k \tag{2.10}$$

The complex variables $\tau = (\tau_0, \tau_1, \cdots, \tau_k)$ are homogeneous coordinates of the complex projective space $\mathbb{CP}^k$, *i.e.*, $\mathbb{C}^{k+1} - \{0, 0, \cdots, 0\}$. The multivalued function Φ is then defined in a space

$$X = \mathbb{CP}^k - \bigcup_{j=0}^{n} \mathcal{H}_j \tag{2.11}$$

where

$$\mathcal{H}_j = \{\tau \in \mathbb{CP}^k ; \; l_j(\tau) = 0\} \tag{2.12}$$

We now consider the meaning of the integral path Δ. Since the integrand $\Phi\omega$ is a multivalued k-form, simple choice of Δ as a k-chain on X is not enough. *Upon the choice of Δ, we need to implicitly specify branches of Φ on Δ as well, otherwise we can not properly define the integral.* In what follows we assume these implicit conditions.

Before considering further properties of Δ, we here notice that ω has an ambiguity in the evaluation of the integral (2.7). Suppose α is an arbitrary $(k-1)$-form defined in X. Then an integral over the exact k-form $d(\Phi\alpha)$ vanishes:

$$0 = \int_\Delta d(\Phi\alpha) = \int_\Delta \Phi \left(d\alpha + \frac{d\Phi}{\Phi} \wedge \alpha \right) = \int_\Delta \Phi \nabla \alpha \tag{2.13}$$

where ∇ can be interpreted as a covariant (exterior) derivative

$$\nabla = d + d \log \Phi \wedge = d + \sum_{j=0}^{n} \alpha_j \frac{dl_j}{l_j} \wedge \tag{2.14}$$

This means that $\omega' = \omega + \nabla\alpha$ is equivalent to ω in the definition of the integral (2.7). Namely, ω and ω' form an equivalent class, $\omega \sim \omega'$. This equivalent class is called the cohomology class.

To study this cohomology class, we consider the differential equation

$$\nabla f = df + \sum_{j=0}^{n} \alpha_j \frac{dl_j}{l_j} f = 0 \tag{2.15}$$

General solutions are locally determined by

$$f = \lambda \prod_{j=0}^{n} l_j(\tau)^{-\alpha_j} \qquad (\lambda \in \mathbb{C}^\times) \tag{2.16}$$

These local solutions are thus basically given by $1/\Phi$. The idea of locality is essential since even if $1/\Phi$ is multivalued within a local patch it can be treated as a single-valued function. Analytic continuation of these solutions forms a fundamental homotopy group of a closed path in X (or $1/X$ to be precise but it can be regarded as X by flipping the non-integer powers α_j in (2.8)). The representation of this fundamental group is called the *monodromy* representation. The monodromy representation determines the local system of the differential equation (2.15). The general solution f or $1/\Phi$ gives a rank-1 local system in this sense[2]. We denote this rank-1 local system by $\mathcal{L}$. The above cohomology class is then defined as an element of the k-th cohomology group of X over $\mathcal{L}$, *i.e.*,

$$[\omega] \in H^k(X, \mathcal{L}) \tag{2.17}$$

This cohomology group $H^k(X, \mathcal{L})$ is also called *twisted* cohomology group.

2.3 Twisted homology and twisted cycles

Having defined the cohomology group $H^k(X, \mathcal{L})$, we can now define the dual of it, *i.e.*, the k-th homology group $H_k(X, \mathcal{L}^\vee)$, known as the twisted homology group, where $\mathcal{L}^\vee$ is the rank-1 dual local system given by Φ. A differential equation corresponding to $\mathcal{L}^\vee$ can be written as

$$\nabla^\vee g \,=\, dg - \sum_{j=0}^{n} \alpha_j \frac{dl_j}{l_j} g \,=\, 0 \tag{2.18}$$

We can easily check that the general solutions are given by Φ:

$$g \,=\, \lambda \prod_{j=0}^{n} l_j(\tau)^{\alpha_j} \,=\, \lambda\Phi \qquad (\lambda \in \mathbb{C}^\times) \tag{2.19}$$

As before, an element of $H_k(X, \mathcal{L}^\vee)$ gives an equivalent class called a homology class.

In the following, we show that the integral path Δ forms an equivalent class and see that it coincides with the above homology class. Applying Stokes' theorem to (2.13), we find

$$0 \,=\, \int_\Delta \Phi \nabla \alpha \,=\, \int_{\partial\Delta} \Phi\alpha \tag{2.20}$$

where α is an arbitrary $(k-1)$-form as before. The boundary operator ∂ is in principle determined from Φ (with information on branches). Denoting $C_p(X, \mathcal{L}^\vee)$ a p-dimensional chain group on X over $\mathcal{L}^\vee$, we can express the boundary operator as $\partial : C_p(X, \mathcal{L}^\vee) \longrightarrow C_{p-1}(X, \mathcal{L}^\vee)$. Since the relation (2.20) holds for an arbitrary α, we find that the k-chain Δ vanishes by the action of ∂:

$$\partial\Delta \,=\, 0 \tag{2.21}$$

The k-chain Δ satisfying above is generically called the k-cycle. In the current framework it is also called the *twisted cycle*. Since the boundary operator satisfies $\partial^2 = 0$, the k-cycle

[2] It is 'rank-1' because each factor $l_j(\tau)$ in the local solutions (2.16) is first order in the elements of τ.

has a redundancy in it. Namely, $\Delta' = \Delta + \partial C_{(+1)}$ also becomes the k-cycle where $C_{(+1)}$ is an arbitrary $(k+1)$-chain or an element of $C_{k+1}(X, \mathcal{L}^\vee)$. Thus Δ and Δ' form an equivalent class, $\Delta \sim \Delta'$, and this is exactly the homology class defined by $H_k(X, \mathcal{L}^\vee)$, *i.e.*,

$$[\Delta] \in H_k(X, \mathcal{L}^\vee) \tag{2.22}$$

To summarize, the generalized hypergeometric function (2.7) is determined by the following bilinear form

$$H_k(X, \mathcal{L}^\vee) \times H^k(X, \mathcal{L}) \longrightarrow \mathbb{C} \tag{2.23}$$

$$([\Delta], [\omega]) \longrightarrow \int_\Delta \Phi\omega \tag{2.24}$$

2.4 Differential equations of $F(Z)$

The condition $l_j(\tau) = 0$ in (2.12) defines a hyperplane in $(k+1)$-dimensional spaces. To avoid redundancy in configuration of hyperplanes, we assume the set of hyperplanes are non-degenerate, that is, we consider the hyperplanes in *general position*. This can be realized by demanding that any $(k+1)$-dimensional minor determinants of the $(k+1) \times (n+1)$ matrix Z are nonzero. We then redefine X in (2.11) as

$$X = \{Z \in Mat_{k+1,n+1}(\mathbb{C}) | \text{ any } (k+1)\text{-dim minor determinants of } Z \text{ are nonzero}\} \tag{2.25}$$

In what follows we implicitly demand this condition in Z. The configuration of $n+1$ hyperplanes in $\mathbb{CP}^k$ is determined by this matrix Z.

Apart from the concept of hyperplanes, we can also interpret that the above Z provides $n+1$ *distinct points* in $\mathbb{CP}^k$. Since a homogeneous coordinate of $\mathbb{CP}^k$ is given by $\mathbb{C}^{k+1} - \{0, 0, \cdots, 0\}$, we can consider each of the $n+1$ column vectors of Z as a point in $\mathbb{CP}^k$; the j-th column representing the j-th homogeneous coordinates of $\mathbb{CP}^k$ ($j = 0, 1, \cdots, n$).

The scale transformation, under which the $\mathbb{CP}^k$ homogeneous coordinates are invariant, is realized by an action of $H_{n+1} = \{\text{diag}(h_0, h_1, \cdots h_n) | h_j \in \mathbb{C}^\times\}$ from right on Z. The general linear transformation of the homogeneous coordinates, on the other hand, can be realized by an action of $GL(k+1, \mathbb{C})$ from left. These transformations are then given by

$$\text{Linear transformation:} \quad Z \to Z' = gZ \tag{2.26}$$

$$\text{Scale transformation:} \quad Z \to Z' = Zh \tag{2.27}$$

where $g \in GL(k+1, \mathbb{C})$ and $h \in H_{n+1}$. Under these transformations the integral $F(Z)$ in (2.7) behaves as

$$F(gZ) = (\det g)^{-1} F(Z) \tag{2.28}$$

$$F(Zh) = F(Z) \prod_{j=0}^{n} h_j^{\alpha_j} \tag{2.29}$$

We now briefly show that the above relations lead to the defining equations of the generalized hypergeometric functions in (2.2) and (2.3), respectively. Let $\mathbf{1}_n$ be the n-dimensional identity matrix $\mathbf{1}_n = \mathrm{diag}(1, 1, \cdots, 1)$, and $E_{ij}^{(n)}$ be an $n \times n$ matrix in which only the (i, j)-element is 1 and the others are zero. We consider g in a particular form of

$$g = \mathbf{1}_{k+1} + \epsilon E_{pi}^{(k+1)} \tag{2.30}$$

where ϵ is a parameter. Then gZ remains the same as Z except the p-th row which is replaced by $(z_{p0} + \epsilon z_{i0}, z_{p1} + \epsilon z_{i1}, \cdots, z_{pn} + \epsilon z_{in})$. Then the derivative of $F(gZ)$ with respect to ϵ is expressed as

$$\frac{\partial}{\partial \epsilon} F(gZ) = \sum_{j=0}^{n} z_{ij} \frac{\partial}{\partial z_{pj}} F(gZ) \tag{2.31}$$

On the other hand, using

$$\mathrm{det} g = \begin{cases} 1 & (i \neq p) \\ \epsilon & (i = p) \end{cases} \tag{2.32}$$

and (2.28), we find

$$\frac{\partial}{\partial \epsilon} F(gZ) = \begin{cases} 0 & (i \neq p) \\ -\frac{1}{\epsilon^2} F(Z) & (i = p) \end{cases} \tag{2.33}$$

Evaluating the derivative at $\epsilon = 0$ and $\epsilon = 1$ for $i \neq p$ and $i = p$, respectively, we then indeed find that (2.28) leads to the differential equation (2.2).

Similarly, parametrizing h as

$$h = \mathrm{diag}(h_0, \cdots, h_{j-1}, (1 + \epsilon) h_j, h_{j+1}, \cdots, h_n) \tag{2.34}$$

with $0 \leq j \leq n$, we find that Zh has only one ϵ-dependent column corresponding to the j-th column, $(z_{0j}(1 + \epsilon)h_j, z_{1j}(1 + \epsilon)h_j, \cdots, z_{kj}(1 + \epsilon)h_j)^T$. The derivative of $F(Zh)$ with respect to ϵ is then expressed as

$$\frac{\partial}{\partial \epsilon} F(Zh) = \sum_{i=0}^{k} z_{ij} \frac{\partial}{\partial z_{ij}} F(Zh) = \sum_{i=0}^{k} z_{ij} \frac{\partial}{\partial z_{ij}} F(Z)(1 + \epsilon)^{\alpha_j} \prod_{l=0}^{n} h_l^{\alpha_l} \tag{2.35}$$

where in the last step we use the relation from (2.29):

$$F(Zh) = F(Z)(1 + \epsilon)^{\alpha_j} \prod_{l=0}^{n} h_l^{\alpha_l} \tag{2.36}$$

The same derivative can then be expressed as

$$\frac{\partial}{\partial \epsilon} F(Zh) = \alpha_j F(Z)(1 + \epsilon)^{\alpha_j - 1} \prod_{l=0}^{k} h_l^{\alpha_l} \tag{2.37}$$

Setting $\epsilon = 0$, we can therefore derive the equation (2.3).

The other equation (2.4) for $F(Z)$ follows from the definition of Φ. From (2.8) and (2.9) we find that Φ satisfies

$$\frac{\partial \Phi}{\partial z_{ip}} = \frac{\alpha_i \tau_p}{l_i(\tau)} \Phi \tag{2.38}$$

This relation leads to

$$\frac{\partial^2 \Phi}{\partial z_{ip} \partial z_{jq}} = \frac{\alpha_i \alpha_j \tau_p \tau_q}{l_i(\tau) l_j(\tau)} \Phi = \frac{\partial^2 \Phi}{\partial z_{iq} \partial z_{jp}} \tag{2.39}$$

which automatically derives the equation (2.4).

The integral $F(Z)$ in (2.7) therefore indeed satisfies the defining equations (2.2)-(2.4) of the generalized hypergeometric functions on $Gr(k+1, n+1)$. *The Grassmannian space* $Gr(k+1, n+1)$ *is defined as a set of* $(k+1)$*-dimensional linear subspaces in* $(n+1)$*-dimensional complex vector space* $\mathbb{C}^{n+1}$. It is defined as

$$Gr(k+1, n+1) = \widetilde{Z}/GL(k+1, \mathbb{C}) \tag{2.40}$$

where $\widetilde{Z}$ is $(k+1) \times (n+1)$ complex matrices with $rank\widetilde{Z} = k+1$. Consider some matrix M and assume that there exists a nonzero r-dimensional minor determinant of M. Then the rank of M is in general defined by the largest number of such r's. Thus $\widetilde{Z}$ is not exactly same as Z defined in (2.25). $\widetilde{Z}$ is more relaxed since it allows some $(k+1)$-dimensional minor determinants vanish, that is, $Z \subseteq \widetilde{Z}$. In this sense $F(Z)$ is conventionally called the generalized hypergeometric functions on $Gr(k+1, n+1)$ and we follow this convention in the present chapter.

2.5 Non-projected formulation

In terms of the homogeneous coordinate $\tau = (\tau_0, \tau_1, \cdots, \tau_k)$ on $\mathbb{CP}^k$, coordinates on $\mathbb{C}^k$ can be parametrized as

$$t_1 = \frac{\tau_1}{\tau_0} \, , \; t_1 = \frac{\tau_1}{\tau_0} \, , \; \cdots \, , \; t_k = \frac{\tau_k}{\tau_0} \tag{2.41}$$

For simplicity, we now fix $(z_{00}, z_{10}, \cdots, z_{n0})^T$ at $(1, 0, \cdots, 0)^T$, *i.e.*,

$$Z = \begin{pmatrix} 1 & z_{01} & z_{02} & \cdots & z_{0n} \\ 0 & z_{11} & z_{12} & \cdots & z_{1n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & z_{k1} & z_{k2} & \cdots & z_{kn} \end{pmatrix} \tag{2.42}$$

Then the integrand of $F(Z)$ can be expressed as

$$\begin{aligned}\Phi\omega &= \tau_0^{\alpha_0}\prod_{j=1}^{n}(\tau_0 z_{0j}+\tau_1 z_{1j}+\cdots+\tau_k z_{kj})^{\alpha_j}\\ &\quad\times\sum_{i=0}^{k}(-1)^i\tau_i d\tau_0\wedge d\tau_1\wedge\cdots\wedge d\tau_{i-1}\wedge d\tau_{i+1}\wedge\cdots\wedge d\tau_k\\ &= \prod_{j=1}^{n}\left(z_{0j}+\frac{\tau_1}{\tau_0}z_{1j}+\cdots+\frac{\tau_k}{\tau_0}z_{kj}\right)^{\alpha_j} d\left(\frac{\tau_1}{\tau_0}\right)\wedge d\left(\frac{\tau_2}{\tau_0}\right)\wedge\cdots\wedge d\left(\frac{\tau_k}{\tau_0}\right)\\ &= \widetilde{\Phi}\widetilde{\omega} \qquad (2.43)\end{aligned}$$

where we use (2.6) and define $\widetilde{\Phi}$, $\widetilde{\omega}$ by

$$\widetilde{\Phi} = \prod_{j=1}^{n}\widetilde{l}_j(t)^{\alpha_j} \qquad (2.44)$$

$$\widetilde{l}_j(t) = z_{0j}+t_1 z_{1j}+t_2 z_{2j}+\cdots+t_k z_{kj} \qquad (1\le j\le n) \qquad (2.45)$$

$$\widetilde{\omega} = dt_1\wedge dt_2\wedge\cdots\wedge dt_k \qquad (2.46)$$

The exponents α_j $(j=1,2,\cdots,n)$ are also imposed to the non-integer conditions $\alpha_j\notin\mathbb{Z}$ and $\alpha_1+\alpha_2+\cdots+\alpha_n\notin\mathbb{Z}$. The multivalued function $\widetilde{\Phi}$ is now defined in the following space

$$\widetilde{X} = \mathbb{C}^k-\bigcup_{j=1}^{n}\widetilde{\mathcal{H}}_j \qquad (2.47)$$

where

$$\widetilde{\mathcal{H}}_j = \{t\in\mathbb{C}^k;\ \widetilde{l}_j(t)=0\} \qquad (2.48)$$

These are non-projected versions of (2.11) and (2.12).

As before, from $\widetilde{\Phi}$ we can define rank-1 local systems $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{L}}^\vee$ on $\widetilde{X}$, which lead to the k-th homology and cohomology groups, $H_k(\widetilde{X},\widetilde{\mathcal{L}}^\vee)$ and $H^k(\widetilde{X},\widetilde{\mathcal{L}})$. Then the integral over $\widetilde{\Phi}\widetilde{\omega}$ is defined as

$$F(Z) = \int_{\widetilde{\Delta}}\widetilde{\Phi}\widetilde{\omega} \qquad (2.49)$$

where $[\widetilde{\Delta}]=H_k(\widetilde{X},\widetilde{\mathcal{L}}^\vee)$ and $[\widetilde{\omega}]=H^k(\widetilde{X},\widetilde{\mathcal{L}})$.

In regard to the cohomology group $H^k(\widetilde{X},\widetilde{\mathcal{L}})$, Aomoto shows the following theorem[3]:

1. The dimension of $H^k(\widetilde{X},\widetilde{\mathcal{L}})$ is given by $\binom{n-1}{k}$.

2. The basis of $H^k(\widetilde{X},\widetilde{\mathcal{L}})$ can be formed by $d\log\widetilde{l}_{j_1}\wedge d\log\widetilde{l}_{j_1}\wedge\cdots\wedge d\log\widetilde{l}_{j_k}$ where $1\le j_1<j_2<\cdots<j_k\le n-1$.

[3]Theorem 9.6.2 in [8]

Correspondingly, the homology group $H_k(\widetilde{X}, \widetilde{\mathcal{L}}^\vee)$ has dimension $\begin{pmatrix} n-1 \\ k \end{pmatrix}$ and its basis can be formed finite regions bounded by $\widetilde{\mathcal{H}}_j$. In terms of $\widetilde{l}_j$'s the basis of $H^k(\widetilde{X}, \widetilde{\mathcal{L}})$ can also be chosen as [9]:

$$\varphi_{j_1 j_2 \cdots j_k} \, = \, d\log \frac{\widetilde{l}_{j_1+1}}{\widetilde{l}_{j_1}} \wedge d\log \frac{\widetilde{l}_{j_2+1}}{\widetilde{l}_{j_2}} \wedge \cdots \wedge d\log \frac{\widetilde{l}_{j_k+1}}{\widetilde{l}_{j_k}} \tag{2.50}$$

where $1 \leq j_1 < j_2 < \cdots < j_k \leq n-1$.

3 Generalized hypergeometric functions on $Gr(2, n+1)$

In this section we consider a particular case of $Gr(2, n+1)$. The corresponding configuration space is simply given by $n+1$ distinct points in $\mathbb{CP}^1$. This can be represented by a $2 \times (n+1)$ matrix Z any of whose 2-dimensional minor determinants are nonzero. Allowing the freedom of the coordinate transformations $GL(2, \mathbb{C})$ from the left and the scale transformations $H_2 = \mathrm{diag}(h_0, h_1)$ from the right, we can uniquely parametrize Z as

$$Z = \begin{pmatrix} 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & -1 & -z_3 & \cdots & -z_n \end{pmatrix} \tag{3.1}$$

where $z_i \neq 0, 1, z_j$ $(i \neq j,\, 3 \leq i, j \leq n)$. Thus we can regard Z as

$$Z \, \simeq \, \{(z_3, z_4, \cdots, z_n) \in \mathbb{C}^{n-2} \,|\, z_i \neq 0, 1, z_j \;\, (i \neq j)\} \tag{3.2}$$

The three other points (z_0, z_1, z_2) can be fixed at $\{0, 1, \infty\}$ This agrees with the fact that the $GL(2, \mathbb{C})$ invariance fixes three points out of the $(n+1)$ distinct points in $\mathbb{CP}^1$.

In application of the previous section, we can carry out a systematic formulation of the generalized hypergeometric functions on $Gr(2, n+1)$ as follows. We begin with a multivalued function of a form

$$\Phi = 1^{\alpha_0} \cdot t^{\alpha_1}(1-t)^{\alpha_2}(1-z_3 t)^{\alpha_3} \cdots (1-z_n t)^{\alpha_n} \, = \, \prod_{j=1}^{n} l_j(t)^{\alpha_j} \tag{3.3}$$

where

$$l_0(t) = 1\,,\;\; l_1(t) = t\,,\;\; l_2(t) = 1-t\,,\;\; l_j(t) = 1 - z_j t \;\;\, (3 \leq j \leq n) \tag{3.4}$$

As in (2.5) and (2.6), the exponents obey the non-integer conditions

$$\alpha_j \notin \mathbb{Z} \;\; (0 \leq j \leq n)\,, \qquad \sum_{j=0}^{n} \alpha_j = -2 \tag{3.5}$$

As considered before, the latter condition applies to the expressions (2.46)-(2.46), that is, when $F(Z)$ is expressed as $F(Z) = \int_\Delta \Phi dt$. The defining space of Φ is given by

$$X = \mathbb{CP}^1 - \{0, 1, 1/z_3, \cdots, 1/z_n, \infty\} \tag{3.6}$$

From Φ we can determine a rank-1 local system $\mathcal{L}$ on X and its dual local system $\mathcal{L}^\vee$. Applying the result in (2.50), the basis of the cohomology group $H^1(X, \mathcal{L})$ is then given by

$$d \log \frac{l_{j+1}}{l_j} \qquad (0 \le j \le n-1) \tag{3.7}$$

In the present case the basis of the homology group $H_1(X, \mathcal{L}^\vee)$ can be specified by a set of paths connecting the branch points. For example, we can choose these by

$$\Delta_{\infty 0}\, ,\ \Delta_{01}\, ,\ \Delta_{1 \frac{1}{z_3}}\, ,\ \Delta_{\frac{1}{z_3} \frac{1}{z_4}}\, ,\ \cdots\, ,\ \Delta_{\frac{1}{z_{n-1}} \frac{1}{z_n}} \tag{3.8}$$

where Δ_{pq} denotes a path on $\mathbb{CP}^1$ connecting branch points p and q. To summarize, for an element $\Delta \in H_1(X, \mathcal{L}^\vee)$ associated with Φ of (3.3), we can define a set of generalized hypergeometric functions on $Gr(2, n+1)$ as

$$f_j(Z) \;=\; \int_\Delta \Phi \, d \log \frac{l_{j+1}}{l_j} \tag{3.9}$$

where $0 \le j \le n-1$. In the next section we consider the case of $n = 3$, the simplest case where only one variable exists, which corresponds to Gauss' hypergeometric function.

4 Reduction to Gauss' hypergeometric function

4.1 Basics of Gauss' hypergeometric function

We first review the basics of Gauss' hypergeometric function. In power series, it is defined as

$$F(a, b, c; z) \;=\; \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n \tag{4.1}$$

where $|z| < 1$, $c \notin \mathbb{Z}_{\le 0}$ and

$$(a)_n = \begin{cases} 1 & (n = 1) \\ a(a+1)(a+2) \cdots (a+n-1) & (n \ge 1) \end{cases} \tag{4.2}$$

$F(a, b, c; z)$ satisfies the hypergeometric differential equation

$$\left[\frac{d^2}{dz^2} + \left(\frac{c}{z} + \frac{a+b+1-c}{z-1} \right) \frac{d}{dz} + \frac{ab}{z(z-1)} \right] F(a, b, c; z) \;=\; 0 \tag{4.3}$$

Euler's integral formula for $F(a, b, c; z)$ is written as

$$F(a, b, c; z) \;=\; \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} (1-zt)^{-b} \, dt \tag{4.4}$$

where $|z| < 1$ and $0 < \Re(a) < \Re(c)$[4]. $\Gamma(a)$'s denote the Gamma functions

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt \qquad (\Re(a) > 0) \tag{4.5}$$

The second order differential equation (4.3) has regular singularities at $z = 0, 1, \infty$. Two independent solutions around each singular point are expressed as

$$z = 0 \ : \quad \begin{cases} f_1(z) = F(a, b, c; z) \\ f_2(z) = z^{1-c} F(a - c + 1, b - c + 1, 2 - c; z) \end{cases} \tag{4.6}$$

$$z = 1 \ : \quad \begin{cases} f_3(z) = F(a, b, a + b - c + 1; 1 - z) \\ f_4(z) = (1 - z)^{c-a-b} F(c - a, c - a, c - a - b + 1; 1 - z) \end{cases} \tag{4.7}$$

$$z = \infty \ : \quad \begin{cases} f_5(z) = z^{-a} F(a, a - c + 1, a - b + 1; 1/z) \\ f_6(z) = z^{-b} F(b - c + 1, b, b - a + 1; 1z) \end{cases} \tag{4.8}$$

where we assume $c \notin \mathbb{Z}$, $a + b - c \notin \mathbb{Z}$ and $a - b \notin \mathbb{Z}$ at $z = 0$, $z = 1$ and $z = \infty$, respectively.

4.2 Reduction to Gauss' hypergeometric function 1: From defining equations

From (4.4) we find the relevant 2×4 matrix in a form of

$$Z = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -z \end{pmatrix} \tag{4.9}$$

The set of equations (2.2)-(2.4) then reduce to the followings:

$$(\partial_{00} + \partial_{02} + \partial_{03})F = -F \tag{4.10}$$
$$(\partial_{11} - \partial_{12} + z\partial_z)F = -F \tag{4.11}$$
$$(\partial_{10} + \partial_{12} - \partial_z)F = 0 \tag{4.12}$$
$$(\partial_{01} + \partial_{02} - \partial_{03})F = 0 \tag{4.13}$$
$$\partial_{00}F = \alpha_0 F \tag{4.14}$$
$$\partial_{11}F = \alpha_1 F \tag{4.15}$$
$$(\partial_{02} - \partial_{12})F = \alpha_2 F \tag{4.16}$$
$$(\partial_{03} + z\partial_z)F = \alpha_3 F \tag{4.17}$$
$$-\partial_z \partial_{02} F = \partial_{12}\partial_{03} F \tag{4.18}$$

where $\partial_{ij} = \frac{\partial}{\partial z_{ij}}$ and $\partial_{13} = -\frac{\partial}{\partial z} = -\partial_z$. The last relation (4.18) arises from (2.4); we here write down the one that is nontrivial and involves ∂_z. Since the sum of (4.10) and (4.11) equals to the sum of (4.14)-(4.17), we can easily find $\alpha_0 + \cdots + \alpha_3 = -2$ in accord with (3.5). The second order equation (4.18) is then expressed as

$$-\partial_z(\alpha_1 + \alpha_2 + 1 + z\partial_z)F = (\alpha_1 + 1 + z\partial_z)(\alpha_3 - z\partial_z)F \tag{4.19}$$

[4]This condition can be relaxed to $a \notin \mathbb{Z}$, $c - a \notin \mathbb{Z}$ by use of the well-known Pochhammer contour in the integral (4.4).

This can also be written as

$$\left[z(1-z)\partial_z^2 + (c-(a+b+1)z)\,\partial_z - ab\right] F \;=\; 0 \tag{4.20}$$

where

$$\begin{aligned} a &= \alpha_1 + 1 \\ b &= -\alpha_3 \\ c &= \alpha_1 + \alpha_2 + 2 \end{aligned} \tag{4.21}$$

We can easily check that (4.20) identifies with the hypergeometric differential equation (4.3).

As seen in (3.1), there exist multiple complex variables for $n > 3$. In these cases reduction of the defining equations (2.2)-(2.4) can be carried out in principle but, unfortunately, is not as straightforward as the case of $n = 3$.

4.3 Reduction to Gauss' hypergeometric function 2: Use of twisted cohomology

The hypergeometric equation (4.3) is a second order differential equation. Setting $f_1 = F$, $f_2 = \frac{z}{b}\frac{d}{dz}F$, we can express (4.3) in a form of a first order Fuchsian differential equation [8]:

$$\frac{d}{dz}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \left(\frac{A_0}{z} + \frac{A_1}{z-1}\right)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{4.22}$$

where

$$A_0 = \begin{pmatrix} 0 & b \\ 0 & 1-c \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ -a & c-a-b-1 \end{pmatrix} \tag{4.23}$$

Using the results (3.3)-(3.9), we now obtain other first order representations of the hypergeometric differential equation.

Let us start with a *non-projected* multivalued function

$$\Phi \;=\; t^a(1-t)^{c-a}(1-zt)^{-b} \tag{4.24}$$

where

$$a\,,\; c-a\,,\; -b \notin \mathbb{Z} \tag{4.25}$$

Φ is defined on $X = \mathbb{CP}^1 - \{0, 1, 1/z, \infty\}$. From these we can determine a rank-1 local system $\mathcal{L}$ and its dual $\mathcal{L}^\vee$ on X. Then, using (3.7), we can obtain a basis of the cohomology group $H^1(X, \mathcal{L})$ given by the following set

$$\varphi_{\infty 0} \;=\; \frac{dt}{t} \tag{4.26}$$

$$\varphi_{01} \;=\; \frac{dt}{t(1-t)} \tag{4.27}$$

$$\varphi_{1\frac{1}{z}} \;=\; \frac{(z-1)dt}{(1-t)(1-zt)} \tag{4.28}$$

Similarly, from (3.8) a basis of the homology group $H_1(X, \mathcal{L}^\vee)$ is given by

$$\{\Delta_{\infty 0}\,,\ \Delta_{01}\,,\ \Delta_{1\frac{1}{z}}\} \tag{4.29}$$

In terms of these we can express Gauss' hypergeometric function as

$$f_{01}(Z) \ = \ \int_{\Delta_{01}} \Phi\varphi_{01} \ = \ \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}dt \tag{4.30}$$

The derivative of $f_{01}(Z) = f_{01}(z)$ with respect to z is written as

$$\frac{d}{dz}f_{01}(z) \ = \ \frac{d}{dz}\int_{\Delta_{01}} \Phi\varphi_{01} \ = \ \int_{\Delta_{01}} \Phi\,\nabla_z\varphi_{01} \tag{4.31}$$

where

$$\nabla_z \ = \ \partial_z + \partial_z \log\Phi \ = \ \partial_z + \frac{bt}{1-zt} \tag{4.32}$$

Thus the derivative comes down to the computation of $\nabla_z\varphi_{01}$; notice that the choice of a twisted cycle Δ is irrelevant as far as the derivative itself is concerned. In order to make sense of (4.31) we should require $\nabla_z\varphi_{01} \in H^1(X, \mathcal{L})$, that is, it should be represented by a linear combinations of (4.26)-(4.28). There is a caveat here, however. We know that an element of $H^1(X, \mathcal{L})$ forms an equivalent class as discussed earlier; see (2.13) and (2.14). In the present case ($k = 1$), α in (2.13) is a 0-form or a constant. So we can demand

$$d\log\Phi \ = \ a\frac{dt}{t} - (c-a)\frac{dt}{1-t} + b\frac{z\,dt}{1-zt} \ \equiv \ 0 \tag{4.33}$$

in the computation of $\nabla_z\varphi_{01}$. This means that the number of the base elements can be reduced from 3 to 2. Namely, any elements of $H^1(X, \mathcal{L})$ can be expressed by a combinations of an arbitrary pair in (4.26)-(4.28) under the condition (4.33). This explains the numbering discrepancies between (2.50) and (3.7) and agrees with the general result in the previous section that the dimension of the cohomology group is given by $\begin{pmatrix} n-1 \\ k \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2$.

Choosing the pair of $(\varphi_{01}, \varphi_{\infty 0})$, we find

$$\begin{aligned}
\nabla_z\varphi_{\infty 0} \ &= \ \frac{bdt}{1-zt} \\
&\equiv \ \frac{1}{z}\left(-a\frac{dt}{t} + (c-a)\frac{dt}{1-t}\right) \\
&= \ \frac{c-a}{z}\varphi_{01} - \frac{c}{z}\varphi_{\infty 0}
\end{aligned} \tag{4.34}$$

$$\begin{aligned}
\nabla_z\varphi_{01} \ &= \ \nabla_z\left(\varphi_{\infty 0} + \frac{dt}{1-t}\right) \\
&= \ \nabla_z\varphi_{\infty 0} + \frac{b}{1-z}\left(\frac{dt}{1-t} - \frac{dt}{1-zt}\right) \\
&= \ \frac{z}{z-1}\nabla_z\varphi_{\infty 0} - \frac{b}{z-1}(\varphi_{01} - \varphi_{\infty 0}) \\
&\equiv \ \frac{c-a-b}{z-1}\varphi_{01} + \frac{b-c}{z-1}\varphi_{\infty 0}
\end{aligned} \tag{4.35}$$

where notation $\equiv$ means the use of condition (4.33). Using (4.31), we obtain a first order differential equation

$$\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{\infty 0} \end{pmatrix} = \left(\frac{A_0^{(\infty 0)}}{z} + \frac{A_1^{(\infty 0)}}{z-1} \right) \begin{pmatrix} f_{01} \\ f_{\infty 0} \end{pmatrix} \tag{4.36}$$

where

$$A_0^{(\infty 0)} = \begin{pmatrix} 0 & 0 \\ c-a & -c \end{pmatrix}, \quad A_1^{(\infty 0)} = \begin{pmatrix} c-a-b & b-c \\ 0 & 0 \end{pmatrix} \tag{4.37}$$

Solving for f_{01}, we can easily confirm that (4.36) leads to Gauss' hypergeometric differential equation (4.3).

Similarly, for the choice of $(\varphi_{01}, \varphi_{1\frac{1}{z}})$ we find

$$\nabla_z \varphi_{01} = \frac{b}{z-1} \varphi_{1\frac{1}{z}} \tag{4.38}$$

$$\begin{aligned} \nabla_z \varphi_{1\frac{1}{z}} &\equiv \nabla_z \left(-\frac{a}{b}\frac{z-1}{z}\varphi_{01} + \frac{c-a}{b}\frac{z-1}{z}\frac{dt}{(1-t)^2} \right) \\ &\equiv -\frac{a}{z}\varphi_{01} + \left(-\frac{c+1}{z} + \frac{c-a-b+1}{z-1} \right) \varphi_{1\frac{1}{z}} \end{aligned} \tag{4.39}$$

Notice that φ_{01} and $\varphi_{1\frac{1}{z}}$ have the same factor $(1-t)^{-1}$. This factor can be absorbed in the definition of Φ in (4.24). Thus, in applying the derivative formula (4.31), we should replace c by $c-1$. This leads to another first order differential equation

$$\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{1\frac{1}{z}} \end{pmatrix} = \left(\frac{A_0^{(1\frac{1}{z})}}{z} + \frac{A_1^{(1\frac{1}{z})}}{z-1} \right) \begin{pmatrix} f_{01} \\ f_{1\frac{1}{z}} \end{pmatrix} \tag{4.40}$$

where

$$A_0^{(1\frac{1}{z})} = \begin{pmatrix} 0 & 0 \\ -a & -c \end{pmatrix}, \quad A_1^{(1\frac{1}{z})} = \begin{pmatrix} 0 & b \\ 0 & c-a-b \end{pmatrix} \tag{4.41}$$

Solving for f_{01}, we can also check that (4.40) becomes Gauss' hypergeometric differential equation (4.3).

The representations (4.23) and (4.37) are obtained by Aomoto-Kita [8] and Haraoka [9], respectively. The last one (4.41) is not known in the literature as far as the author notices. Along the lines of the above derivation, we can also obtain the Aomoto-Kita representation (4.23) as follows. We introduce a new one-form

$$\widetilde{\varphi}_{1\frac{1}{z}} = \frac{z}{z-1}\varphi_{1\frac{1}{z}} = \frac{z\, dt}{(1-t)(1-zt)} \tag{4.42}$$

The corresponding hypergeometric function is given by $\widetilde{f}_{1\frac{1}{z}} = \int_{\Delta_{01}} \Phi \widetilde{\varphi}_{1\frac{1}{z}}$. From (4.38) we can easily see $\nabla_z \varphi_{01} = \frac{b}{z}\widetilde{\varphi}_{1\frac{1}{z}}$. This is consistent with the condition $f_1 = F$, $f_2 = \frac{z}{b}\frac{d}{dz}F$ in (4.22). Since z is defined as $z \neq 0, 1$, $\frac{b}{z}\widetilde{\varphi}_{1\frac{1}{z}}$ and $\frac{b}{z-1}\varphi_{1\frac{1}{z}}$ are equally well defined one-forms.

We can then choose the pair $(\varphi_{01}, \widetilde{\varphi}_{1\frac{1}{z}})$ as a possible basis of the cohomology group. The derivatives $\nabla_z \varphi_{01}$, $\nabla_z \widetilde{\varphi}_{1\frac{1}{z}}$ are calculated as

$$\begin{aligned} \nabla_z \varphi_{01} &= \frac{b}{z} \widetilde{\varphi}_{1\frac{1}{z}} \qquad (4.43) \\ \nabla_z \widetilde{\varphi}_{1\frac{1}{z}} &\equiv \nabla_z \left(-\frac{a}{b} \varphi_{01} + \frac{c-a}{b} \frac{dt}{(1-t)^2} \right) \\ &\equiv -\frac{a}{z-1} \varphi_{01} + \left(\frac{-c}{z} + \frac{c-a-b}{z-1} \right) \widetilde{\varphi}_{1\frac{1}{z}} \qquad (4.44) \end{aligned}$$

where we use the relations

$$\begin{aligned} \frac{t\,dt}{(1-zt)(1-t)} &= \frac{1}{z-1} \left(\frac{dt}{1-zt} - \frac{dt}{1-t} \right) \qquad (4.45) \\ \frac{dt}{(1-t)^2} &\equiv \frac{1}{c-a} \left(a\varphi_{01} + b\widetilde{\varphi}_{1\frac{1}{z}} \right) \qquad (4.46) \end{aligned}$$

As before, φ_{01} and $\widetilde{\varphi}_{1\frac{1}{z}}$ have the same factor $(1-t)^{-1}$. Thus, replacing c by $c-1$, we obtain a first order differential equation

$$\frac{d}{dz} \begin{pmatrix} f_{01} \\ \widetilde{f}_{1\frac{1}{z}} \end{pmatrix} = \left(\frac{\widetilde{A}_0^{(1\frac{1}{z})}}{z} + \frac{\widetilde{A}_1^{(1\frac{1}{z})}}{z-1} \right) \begin{pmatrix} f_{01} \\ \widetilde{f}_{1\frac{1}{z}} \end{pmatrix} \qquad (4.47)$$

where $\widetilde{f}_{1\frac{1}{z}} = \int_{\Delta_{01}} \Phi \widetilde{\varphi}_{1\frac{1}{z}}$ and

$$\widetilde{A}_0^{(1\frac{1}{z})} = \begin{pmatrix} 0 & b \\ 0 & 1-c \end{pmatrix}, \quad \widetilde{A}_1^{(1\frac{1}{z})} = \begin{pmatrix} 0 & 0 \\ -a & c-a-b-1 \end{pmatrix} \qquad (4.48)$$

We therefore reproduce the Aomoto-Kita representation (4.22), (4.23) by a systematic construction of first order representations of the hypergeometric differential equation.

Lastly, we note that $\varphi_{\infty 0} = \frac{dt}{t}$ and $\varphi_{01} = \frac{dt}{t(1-t)}$ have the same factor t^{-1} but we can not absorb this factor into Φ. This is because we can not obtain dt as a base element of $H^1(X, \mathcal{L})$ which is generically given in a form of $d \log \frac{l_{j+1}}{l_j}$ as discussed in (3.7).

4.4 Reduction to Gauss' hypergeometric function 3: Permutation invariance

The choice of twisted cycles or Δ's is irrelevant in the above derivations of the first order Fuchsian differential equations. The hypergeometric function is therefore satisfied by a more general integral form, rather than (4.30), *i.e.*,

$$f_{01}^{(\Delta_{pq})}(z) = \int_{\Delta_{pq}} \Phi \varphi_{01} = \int_p^q t^{a-1} (1-t)^{c-a-1} (1-zt)^{-b} dt \qquad (4.49)$$

where (p,q) represents an arbitrary pair among the four branch points $p, q \in \{0, 1, 1/z, \infty\}$. This means that we can impose *permutation invariance* on the branch points. Δ_{pq} is then given by the following set of twisted cycles:

$$\Delta_{pq} \;=\; \{\Delta_{\infty 0}\,,\,\Delta_{01}\,,\,\Delta_{1\frac{1}{z}}\,,\,\Delta_{1\infty}\,,\,\Delta_{\frac{1}{z}\infty}\,,\,\Delta_{0\frac{1}{z}}\} \tag{4.50}$$

so that the number of elements becomes $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$. Correspondingly, the base elements of the cohomology group also include

$$\varphi_{1\infty} \;=\; \frac{dt}{1-t} \tag{4.51}$$

$$\varphi_{\frac{1}{z}\infty} \;=\; \frac{z\,dt}{1-zt} \tag{4.52}$$

$$\varphi_{0\frac{1}{z}} \;=\; \frac{dt}{t(1-zt)} \tag{4.53}$$

besides (4.26)-(4.28). It is known that $f_{01}^{(\Delta_{pq})}(z)$ are related to the local solutions $f_i(z)$ $(i = 1, 2, \cdots, 6)$ in (4.6)-(4.8) by

$$f_{01}^{(\Delta_{01})}(z) \;=\; B(a, c-a) f_1(z) \tag{4.54}$$

$$f_{01}^{(\Delta_{\frac{1}{z}\infty})}(z) \;=\; e^{i\pi(a+b-c+1)} B(b-c+1, 1-b) f_2(z) \tag{4.55}$$

$$f_{01}^{(\Delta_{\infty 0})}(z) \;=\; e^{i\pi(1-a)} B(a, b-c+1) f_3(z) \tag{4.56}$$

$$f_{01}^{(\Delta_{1\frac{1}{z}})}(z) \;=\; e^{i\pi(a-c+1)} B(c-a, 1-b) f_4(z) \tag{4.57}$$

$$f_{01}^{(\Delta_{1\frac{1}{z}})}(z) \;=\; B(a, 1-b) f_5(z) \tag{4.58}$$

$$f_{01}^{(\Delta_{1\infty})}(z) \;=\; e^{-i\pi(a+b-c+1)} B(b-c+1, c-a) f_6(z) \tag{4.59}$$

where $B(a,b)$ is the beta function

$$B(a,b) \;=\; \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \;=\; \int_0^1 t^{a-1}(1-t)^{b-1} dt \qquad (\Re(a) > 0\,,\, \Re(b) > 0) \tag{4.60}$$

(For derivations and details of these relations, see [9].)

The relevant configuration space represented by Z is given by $Gr(2,4)/\mathcal{S}_4$ where $\mathcal{S}_4$ denotes the rank-4 symmetry group. The permutation invariance can also be confirmed by deriving another set of the first order differential equations with the choice of φ_{01} and one of (4.51)-(4.53). This is what we will present in the following.

For the choice of $(\varphi_{01}, \varphi_{1\infty})$ we find

$$\begin{aligned} \nabla_z \varphi_{1\infty} \;&=\; \frac{bt}{1-zt} \frac{dt}{(1-t)} \\ &\equiv\; -\frac{a}{z(z-1)} \varphi_{01} + \left(\frac{c}{z(z-1)} - \frac{b}{z-1} \right) \varphi_{1\infty} \end{aligned} \tag{4.61}$$

$$\begin{aligned} \nabla_z \varphi_{01} \;&=\; \nabla_z \frac{dt}{t} + \nabla_z \varphi_{1\infty} \\ &\equiv\; -\frac{a}{z-1} \varphi_{01} + \frac{c-b}{z-1} \varphi_{1\infty} \end{aligned} \tag{4.62}$$

The corresponding differential equation is then expressed as

$$\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{1\infty} \end{pmatrix} = \left(\frac{A_0^{(1\infty)}}{z} + \frac{A_1^{(1\infty)}}{z-1} \right) \begin{pmatrix} f_{01} \\ f_{1\infty} \end{pmatrix} \tag{4.63}$$

where

$$A_0^{(1\infty)} = \begin{pmatrix} 0 & 0 \\ a & -c \end{pmatrix}, \quad A_1^{(1\infty)} = \begin{pmatrix} -a & c-b \\ -a & c-b \end{pmatrix} \tag{4.64}$$

Solving for f_{01}, we can check that (4.63) indeed becomes Gauss' hypergeometric differential equation (4.3).

Similarly, for $(\varphi_{01}, \varphi_{\frac{1}{z}\infty})$ we find

$$\begin{aligned} \nabla_z \varphi_{01} &= \frac{b\,dt}{(1-zt)(1-t)} \\ &\equiv -\frac{1}{z-1}\frac{ab}{c}\varphi_{01} - \frac{1}{z-1}\frac{b}{c}(b-c)\varphi_{\frac{1}{z}\infty} \end{aligned} \tag{4.65}$$

$$\begin{aligned} \nabla_z \varphi_{\frac{1}{z}\infty} &\equiv \nabla_z \left(-\frac{a}{b}\varphi_{01} + \frac{c}{b}\frac{dt}{1-t} \right) \\ &\equiv \frac{1}{z-1}\frac{a}{c}(a-c)\varphi_{01} + \left(\frac{1}{z-1}\frac{1}{c}(b-c)(a-c) - \frac{c}{z} \right) \varphi_{\frac{1}{z}\infty} \end{aligned} \tag{4.66}$$

The first order differential equation is then expressed as

$$\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{\frac{1}{z}\infty} \end{pmatrix} = \left(\frac{A_0^{(\frac{1}{z}\infty)}}{z} + \frac{A_1^{(\frac{1}{z}\infty)}}{z-1} \right) \begin{pmatrix} f_{01} \\ f_{\frac{1}{z}\infty} \end{pmatrix} \tag{4.67}$$

where

$$A_0^{(\frac{1}{z}\infty)} = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}, \quad A_1^{(\frac{1}{z}\infty)} = \begin{pmatrix} -\frac{ab}{c} & -\frac{b}{c}(b-c) \\ \frac{a}{c}(a-c) & \frac{1}{c}(b-c)(a-c) \end{pmatrix} \tag{4.68}$$

We can check that (4.67) reduces to the hypergeometric differential equation for f_{01}.

Lastly, for $(\varphi_{01}, \varphi_{0\frac{1}{z}})$ we find

$$\begin{aligned} \nabla_z \varphi_{01} &= \frac{b\,dt}{(1-zt)(1-t)} \\ &= -\frac{b}{z-1}\left(\varphi_{01} - \varphi_{0\frac{1}{z}} \right) \end{aligned} \tag{4.69}$$

$$\begin{aligned} \nabla_z \varphi_{0\frac{1}{z}} &\equiv \frac{b\,dt}{(1-zt)^2} \\ &\equiv -\frac{c-a}{z(z-1)}\varphi_{01} + \frac{c-az}{z(z-1)}\varphi_{0\frac{1}{z}} \end{aligned} \tag{4.70}$$

The corresponding differential equation becomes

$$\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{0\frac{1}{z}} \end{pmatrix} = \left(\frac{A_0^{(0\frac{1}{z})}}{z} + \frac{A_1^{(0\frac{1}{z})}}{z-1} \right) \begin{pmatrix} f_{01} \\ f_{0\frac{1}{z}} \end{pmatrix} \tag{4.71}$$

where

$$A_0^{(0\frac{1}{z})} = \begin{pmatrix} 0 & 0 \\ c-a & -c \end{pmatrix}, \quad A_1^{(0\frac{1}{z})} = \begin{pmatrix} -b & b \\ -c+a & c-a \end{pmatrix} \tag{4.72}$$

We can check that (4.71) reduces to the hypergeometric differential equation for f_{01} as well.

As in the case of (4.42), it is tempting to think of $\widetilde{\varphi}_{\frac{1}{z}\infty} = \frac{z-1}{z}\varphi_{\frac{1}{z}\infty} = \frac{(z-1)\,dt}{1-zt}$. But, with φ_{01} and $\widetilde{\varphi}_{\frac{1}{z}\infty}$, it is not feasible to obtain a first order differential equation in the form of (4.67) which leads to the hypergeometric differential equation. This is because, if expanded in φ_{01} and $\widetilde{\varphi}_{\frac{1}{z}\infty}$, the z-dependence of the derivatives $\nabla_z\varphi_{01}$ and $\nabla_z\widetilde{\varphi}_{\frac{1}{z}\infty}$, can not be written in terms of $\frac{1}{z}$ or $\frac{1}{z-1}$.

4.5 Summary

In this section we carry out a systematic derivation of first order representations of the hypergeometric differential equation by use of twisted cohomology as the simplest reduction of Aomoto's generalized hypergeometric function. The first order equations are generically expressed as

$$\frac{d}{dz}\begin{pmatrix} f_{01} \\ f_{pq} \end{pmatrix} = \left(\frac{A_0^{(pq)}}{z} + \frac{A_1^{(pq)}}{z-1}\right)\begin{pmatrix} f_{01} \\ f_{pq} \end{pmatrix} = A_{01}^{(pq)}\begin{pmatrix} f_{01} \\ f_{pq} \end{pmatrix} \tag{4.73}$$

where (pq) denotes a pair of four branch points $\{0, 1, 1/z, \infty\}$ in $\Phi = t^a(1-t)^{c-a}(1-zt)^{-b}$. A list of the (2×2) matrices $A_{01}^{(pq)}$ obtained in this section is given by the following:

$$A_{01}^{(\infty 0)} = \begin{pmatrix} \frac{c-a-b}{z-1} & \frac{b-c}{z-1} \\ \frac{c-a}{z} & -\frac{c}{z} \end{pmatrix} \tag{4.74}$$

$$A_{01}^{(1\frac{1}{z})} = \begin{pmatrix} 0 & \frac{b}{z-1} \\ -\frac{a}{z} & -\frac{c}{z} + \frac{c-a-b}{z-1} \end{pmatrix} \tag{4.75}$$

$$\widetilde{A}_{01}^{(1\frac{1}{z})} = \begin{pmatrix} 0 & \frac{b}{z} \\ -\frac{a}{z-1} & -\frac{c-1}{z} + \frac{c-a-b-1}{z-1} \end{pmatrix} \tag{4.76}$$

$$A_{01}^{(1\infty)} = \begin{pmatrix} -\frac{a}{z-1} & \frac{c-b}{z-1} \\ \frac{a}{z} - \frac{a}{z-1} & -\frac{c}{z} + \frac{c-b}{z-1} \end{pmatrix} \tag{4.77}$$

$$A_{01}^{(\frac{1}{z}\infty)} = \begin{pmatrix} -\frac{1}{z-1}\frac{ab}{c} & -\frac{1}{z-1}\frac{b}{c}(b-c) \\ \frac{1}{z-1}\frac{a}{c}(a-c) & -\frac{c}{z} + \frac{1}{z-1}\frac{1}{c}(b-c)(a-c) \end{pmatrix} \tag{4.78}$$

$$A_{01}^{(0\frac{1}{z})} = \begin{pmatrix} -\frac{b}{z-1} & \frac{b}{z-1} \\ \frac{c-a}{z} - \frac{c-a}{z-1} & -\frac{c-1}{z} + \frac{c-a}{z-1} \end{pmatrix} \tag{4.79}$$

where we include the Aomoto-Kita representation $\widetilde{A}_{01}^{(1\frac{1}{z})}$. As far as the author notices, these expressions except (4.74, 4.76) are new for the description of the hypergeometric differential equation. A common feature among these matrices is that the determinant is identical:

$$\det A_{01}^{(pq)} = \frac{ab}{z(z-1)} \tag{4.80}$$

In terms of the first order differential equation (4.73), this means that the action of the derivative on the basis $\begin{pmatrix} f_{01} \\ f_{pq} \end{pmatrix}$ of the cohomology group $H^1(X, \mathcal{L})$ can be represented by a generator of the $SL(2, \mathbb{C})$ algebra. In other words, a change of the bases is governed by the $SL(2, \mathbb{C})$ symmetry. The $SL(2, \mathbb{C})$ invariance corresponds to the global conformal symmetry for holomorphic functions on $\mathbb{CP}^1$. In the present case we start from the holomorphic multivalued function Φ in (4.24) which is defined on $X = \mathbb{CP}^1 - \{0, 1, 1/z, \infty\}$. The result (4.80) is thus natural in concept but nontrivial in practice because the equivalence condition $d \log \Phi \equiv 0$ in (4.33) is implicitly embedded into the expressions (4.74)-(4.79).

To conclude this chapter, we first review the definition of Aomoto's generalized hypergeometric functions on $Gr(k+1, n+1)$, interpreting their integral representations in terms of twisted homology and cohomology. We then consider reduction of the general $Gr(k+1, n+1)$ case to particular $Gr(2, n+1)$ cases. The case of $Gr(2, 4)$ leads to Gauss' hypergeometric functions. We carry out a thorough study of this case in the present section. Much of the present chapter, by nature, deals with reviews of existed literature. But the results in (4.73)-(4.80) are new as far as the author notices.

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