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# The blue sky and the light of setting suns

V.P. NAIR

*Physics Department, City College of the CUNY, New York, NY 10031*

## Abstract

Here are a few thoughts when watching the continuum of the blue sky merging with the blue ocean on Eggremni beach: Is Rayleigh scattering the reason for all the blue? If it is universal, shouldn't there be a symmetry-based low energy theorem? Where do fluctuations fit in? This is an old problem, much has been written on it. I want to think about it from a field theory angle.

Midway through the poem *Lines composed a few miles above Tintern Abbey* we find the beautiful and intriguing lines

..... And I have felt  
A presence that disturbs me with the joy  
Of elevated thoughts; a sense sublime  
Of something far more deeply interfused,  
Whose dwelling is the light of setting suns,  
And the round ocean, and the living air,  
And the blue sky and in the mind of man,  
.....

Perhaps William Wordsworth, for whom nature was indeed "deeply interfused" with his very being, intuited something deeper than the visual joy of sunsets and the blue sky. A physicist cannot but be struck by the aptness of his lines, and as unpacking of this mystery, or the "unweaving of this rainbow" as Richard Dawkins has argued so eloquently [1], can only deepen our appreciation and not subtract from the wonder of it, let us dwell on it a while.

# 1 Rayleigh scattering: the basics

We can understand the blueness of the sky and the orange-red glow of sunsets in terms of the scattering of light by the molecules in air. A small part of the explanation of the blue color of the ocean is in scattering as well.

Leonardo da Vinci seems to have been first, at least in the body of as-yet-discovered records of the past, to suggest that the blueness of the sky may be due to scattering of light. He noticed that smoke, seen against light, acquires a blue tint and argued that a similar phenomenon would explain the blue sky. In the nineteenth century, Tyndall carried out controlled experiments on the scattering of light from colloidal suspensions confirming how the blue color could arise. The theoretical explanation started with Rayleigh who calculated the scattering of electromagnetic radiation from dielectric materials and atoms and obtained the  $\omega^4$ -dependence of the cross section, where  $\omega$  is the frequency of the light. This frequency dependence leads to a dispersion in which the blue component of the solar spectrum can be more scattered and dispersed in the atmosphere (compared to the red end of the spectrum) giving rise to a diffuse blue color. The lower frequencies or the longer wavelengths scatter less and propagate more directly to the observer on Earth. However, as pointed out by Smoluchowski in 1909, the amplitude for scattering from a truly random distribution of scatterers is zero. So while Rayleigh's calculation of the scattering cross section is



Figure 1: Egremni beach, Lefkada, Greece

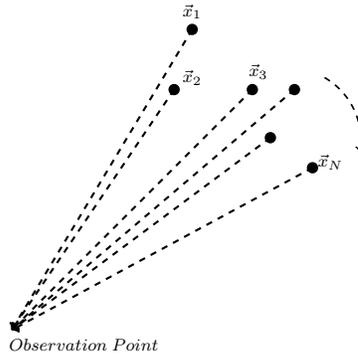


Figure 2: Calculating amplitude by Huyghens' principle

correct (and brilliant), its application seemingly would fail if the atmosphere is viewed as truly random distribution of molecules with uniform density. Einstein and Smoluchowski showed that scattering requires density fluctuations in the atmosphere - the "living air" with density fluctuations is needed. This reasoning also then leads to the explanation for critical opalescence [2].

Let us start with a simple minded way of understanding the problem raised by Smoluchowski. For this, consider a collection of  $N$  molecules or scattering centers, at various positions, say,  $x_1, x_2, \dots, x_N$ . The observation is done at some point which we may take as the origin of coordinates, see Fig. 2. By Huyghens' principle, we may take each scattering center as the source of spherical waves which propagate to the observation point which have to be summed up to get the observed amplitude. Thus, we can write this amplitude as

$$\mathcal{A} \propto \sum_i \frac{e^{ik|\vec{x}_i|}}{|\vec{x}_i|} \quad (1)$$

Let  $\vec{x}$  denote a point in the cluster of molecules, like a center of mass position, then  $\vec{x}_i = \vec{x} + \xi_i$ . If the observation point is sufficiently far away from the molecules, we can approximate  $|\vec{x}_i| \approx |\vec{x}|$  in the denominator of (1). Then we get

$$\mathcal{A} \propto \frac{e^{ik|\vec{x}|}}{|\vec{x}|} \sum_i e^{ik\hat{x}\cdot\xi_i} \quad (2)$$

For a random distribution of particles at positions  $\xi_i$ , the phases  $e^{ik\hat{x}\cdot\xi_i}$  are random and hence the sum gives zero. This is the essence of Smoluchowski's observation. We can do a more serious calculation which will reproduce this result and also show the way out of this conclusion. But first, let us calculate the scattering cross section for photons which will show the  $\omega^4$ -dependence; later we can rederive the same results from more general symmetry arguments.

For scattering from neutral composites of charges, such as atoms and molecules, we have  $\int d^3x J^0 = 0$  (since the total charge is zero) and further, given the conservation equation  $\partial_\alpha J^\alpha = 0$ , we can write the current as

$$J_\mu = \partial_\alpha M^{\alpha\mu} \quad (3)$$

The antisymmetric tensor  $M^{\alpha\mu}$  has components  $M^{0i} = p^i$ ,  $M^{ij} = \epsilon^{ijk} m^k$ ;  $p^i$  and  $m^k$  are the electric and magnetic dipole moment operators, respectively. Notice that (3) gives  $J^0 = -\nabla \cdot \vec{p}$  so that we will have  $\int d^3x J^0 = 0$  for  $\vec{p}$  vanishing at spatial infinity. Thus overall neutrality is needed for us to be able to represent the current as in (3). The electromagnetic interaction term may be simplified in this case as

$$\begin{aligned} \int d^4x J^\mu A_\mu &= -\frac{1}{2} \int d^4x F_{\mu\nu} M^{\mu\nu} \\ &= - \int d^4x (\vec{p} \cdot \vec{E} + \vec{m} \cdot \vec{B}) \end{aligned} \quad (4)$$

For the scattering of low energy photons (of energies around the visible range) from atoms or molecules, it is adequate to keep just the electric dipole term.

The interaction term leads to the scattering operator or the S-operator

$$\hat{S} = T e^{i \int d^4x J^\mu A_\mu} \quad (5)$$

where  $T$  signifies the time-ordering of the operator to the right of it. The amplitude for the scattering of a photon of momentum  $\vec{k}$ , polarization  $\epsilon_i$  to a photon of momentum  $\vec{k}'$ , polarization  $\epsilon'_j$ , to the lowest order in perturbation theory, is thus given by

$$A = - \int \langle f | T p^i(x) p^j(y) | i \rangle \omega \omega' \frac{\epsilon_i e^{-ikx}}{\sqrt{2\omega V}} \frac{\epsilon'_j e^{-ik'x}}{\sqrt{2\omega' V}} \quad (6)$$

where  $\omega = \omega_k$ ,  $\omega' = \omega_{k'}$  and  $T$  denotes time-ordering as usual. Further,  $|i\rangle$  denotes the initial state of the medium,  $|f\rangle$  denotes the final state. The initial state is chosen out of a statistical distribution of states characterized by a density matrix  $\rho$ .

For scattering from atoms and molecules for photon energies small compared to the nuclear masses, the motion of the nuclei induced by the electromagnetic field of the photon can be neglected and the dipole operator can be approximated by  $p_i(x) = e\psi^\dagger(x) \xi_i \psi(x)$  where  $\xi$  is the position of the electron relative to the nucleus. The initial state consists of a number of scattering centers at positions  $\vec{a}_n$  (with the electrons around them), so that we may write

$$|i\rangle = \int [d\xi_n] \psi^\dagger(a_n + \xi_n) f_{\alpha_n}(\xi_n) |0\rangle \quad (7)$$

where  $f_{\alpha_n}(\xi_n)$  is the electron wave function in the  $n$ -th atom or molecule, in the electronic state  $|\alpha_n\rangle$ . Thus, for  $x^0 > y^0$ , we can write

$$\int e^{-ikx+ik'y} \langle f | T p_i(x) p_j(y) | y \rangle$$

$$\begin{aligned}
&= e^2 \int dx^0 \int_{-\infty}^{x^0} dy^0 \int e^{-ikx+ik'y} \langle f | \psi^\dagger(x) \xi_i \psi(x) \psi^\dagger(y) \xi_j \psi(y) | i \rangle \\
&= e^2 \int dx^0 \int_{-\infty}^{x^0} dy^0 \int e^{-ikx+ik'y} e^{-iE_{\alpha_n} y^0} \delta^{(3)}(y - a_n - \xi_n) \\
&\quad \times \langle f | \psi^\dagger(x) \xi_i \psi(x) \psi^\dagger(y) \xi_j f_{\alpha_n}(y) | i \rangle
\end{aligned} \tag{8}$$

In simplifying this further, we notice that there are, in principle, terms where the atom at  $\vec{a}_n$  is excited by the incoming photon but a different atom at  $\vec{a}_m$  is de-excited by the emission of the outgoing photon. Such processes are negligible at low temperatures<sup>1</sup> where it is unlikely to have atom at  $\vec{a}_m$  excited to begin with. In this case we may take the scattering, both the absorption and the emission of the photon, to be due to the same atom. This leads to the simplification of (8) as

$$\begin{aligned}
&\int e^{-ikx+ik'y} \langle f | T p_i(x) p_j(y) | y \rangle \\
&= e^2 \sum_n \int dx^0 \int_{-\infty}^{x^0} dy^0 \delta^{(3)}(y - \vec{a}_n - \xi_n) \delta^{(3)}(x - \vec{a}_n - \xi'_n) e^{-iE_{\alpha_n} y^0 + iE_{\beta_n} y^0} \\
&\quad \times \sum_{\beta} [f_{\gamma_n}^* \xi_n f_{\beta_n}] (x) [f_{\beta_n}^* \xi_j f_{\alpha_n}] (y) e^{-iE_{\beta_n} x^0 + iE_{\gamma_n} x^0} \\
&= \sum_{n,\beta} e^{i(\vec{k}-\vec{k}') \cdot \vec{a}_n} 2\pi \delta(E_\alpha + \omega - E_\gamma - \omega') (-ie^2) \frac{\langle \gamma | e^{i\vec{k} \cdot \xi} \xi_i | \beta \rangle \langle \beta | e^{-i\vec{k}' \cdot \xi} \xi_j | \alpha \rangle}{(E_\beta - E_\alpha + \omega' - i\epsilon)}
\end{aligned} \tag{9}$$

The matrix element refers to scattering at a single atom. We have taken the scatterers at different points to be of the same type, so that the matrix element has the same form. Notice that we can write  $\sum_n e^{i(\vec{k}-\vec{k}') \cdot \vec{a}_n} = \int d^3x e^{i(\vec{k}-\vec{k}') \cdot \vec{x}} N(x)$  where  $N(x) = \sum_n \delta^{(3)}(x - a_n)$  is the density of the scatterers. We can thus write the result as

$$\begin{aligned}
&\int e^{-ikx+ik'y} \langle f | T p_i(x) p_j(y) | y \rangle \\
&= \int d^3x e^{i(\vec{k}-\vec{k}') \cdot \vec{x}} N(x) 2\pi \delta(E_\alpha + \omega - E_\gamma - \omega') (\mathcal{M}_{\gamma\alpha})_{ij}
\end{aligned} \tag{10}$$

where the matrix element, taking account of both time-orderings, is given by

$$(\mathcal{M}_{\gamma\alpha})_{ij} = -ie^2 \sum_{\beta} \left[ \frac{\langle \gamma | e^{i\vec{k} \cdot \xi} \xi_i | \beta \rangle \langle \beta | e^{-i\vec{k}' \cdot \xi} \xi_j | \alpha \rangle}{(E_\beta - E_\alpha + \omega' - i\epsilon)} + \frac{\langle \gamma | e^{-i\vec{k}' \cdot \xi} \xi_j | \beta \rangle \langle \beta | e^{i\vec{k} \cdot \xi} \xi_i | \alpha \rangle}{(E_\beta - E_\alpha - \omega - i\epsilon)} \right] \tag{11}$$

We use this result in (6), take the square of the amplitude and sum over a set of final states for the photon and all final states for the medium to get the transition rate as

$$\frac{\sum |\mathcal{A}|^2}{\tau} = \frac{1}{4} \int \frac{d^3k'}{(2\pi)^3} \frac{\omega \omega' \epsilon_i \epsilon_r}{V} P_{js}(k') \int d^3x d^3y e^{i(\vec{k}-\vec{k}') \cdot (\vec{x}-\vec{y})} N(x) N(y)$$

<sup>1</sup>low compared to the energies of the photons under consideration

$$\times (\mathcal{M}_{\gamma\alpha})_{ij} (\mathcal{M}_{\gamma\alpha}^*)_{rs} 2\pi\delta(E_\alpha + \omega - E_\gamma - \omega') \quad (12)$$

where

$$P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{\omega^2} \quad (13)$$

We now average over the initial polarizations of the photon, since the incident light is unpolarized, and average over the initial states with the density matrix  $\rho$ , to obtain the unpolarized differential cross section (averaged over the statistical distribution) as

$$d\sigma = \frac{1}{8} \int \frac{d^3 k'}{(2\pi)^3} \omega \omega' P_{ir}(k) P_{js}(k') 2\pi\delta(E_\alpha + \omega - E_\gamma - \omega') \\ \times \int d^3 x d^3 y e^{i(\vec{k}-\vec{k}')\cdot(\vec{x}-\vec{y})} \text{Tr} [\rho N(x) N(y) (\mathcal{M}_{\gamma\alpha})_{ij} (\mathcal{M}_{\gamma\alpha}^*)_{rs}] \quad (14)$$

The trace over the density matrix leads to statistically averaged correlation function for densities; it may also involve a possible trace over the initial states  $|\alpha\rangle$ , with some weight factors  $c_\alpha$ , if there could be several initial states for the atom due to some degree of thermal excitation, for instance. Separating these out, the cross section can be written as

$$d\sigma = S(\vec{k} - \vec{k}') d\sigma_{atom} \\ d\sigma_{atom} = \frac{1}{8} \int \frac{d^3 k'}{(2\pi)^3} \omega \omega' P_{ir}(k) P_{js}(k') 2\pi\delta(E_\alpha + \omega - E_\gamma - \omega') \\ \times \sum_{\alpha\gamma} c_\alpha (\mathcal{M}_{\alpha\gamma}^\dagger)_{rs} (\mathcal{M}_{\gamma\alpha})_{ij} \quad (15)$$

In this equation,  $S(\vec{k} - \vec{k}')$  is the structure factor or the Fourier transform of the density correlation function; explicitly,

$$S(\vec{k} - \vec{k}') = \int d^3 x d^3 y e^{i(\vec{k}-\vec{k}')\cdot(\vec{x}-\vec{y})} \text{Tr}[\rho N(x) N(y)] \quad (16)$$

$d\sigma_{atom}$  denotes the cross section for scattering from a single atom. The integration over  $dk' = d\omega'$  can be carried out, and this sets  $\omega' = \omega + E_\alpha - E_\gamma$ . In the case of elastic scattering, where the final state of the atom is not an excited state, we have  $E_\alpha = E_\gamma$ , and  $\omega' = \omega$ . The cross section for a single atom is then

$$\frac{d\sigma_{atom}}{d\Omega} \approx \frac{\omega^4}{32\pi^2} P_{ir}(k) P_{js}(k') \sum_{\alpha\gamma} c_\alpha (\mathcal{M}_{\alpha\gamma}^\dagger)_{rs} (\mathcal{M}_{\gamma\alpha})_{ij} \quad (17)$$

This shows the characteristic  $\omega^4$ -dependence. (The matrix elements also contain terms involving  $\omega$ , but this  $\omega$ -dependence is again not important at low energies; the  $\omega^4$ -dependence is not altered.)

In (16), if  $N(x)$  is constant, then we have  $S(\vec{k} - \vec{k}') \sim \delta^{(3)}(k - k')$ , and there is no scattering, only propagation in the forward direction. This is essentially the observation

made by Smoluchowski that there would be no scattering without density fluctuations. To see how nontrivial scattering arises, we can calculate the density correlation function for a simple case, such as a gas of nonrelativistic atoms. We take  $N(x) = \phi^\dagger \phi$ , where  $\phi$  denotes the annihilation operator for the atom. The disconnected contribution  $\langle N(x) \rangle \langle N(y) \rangle$  leads to the constant density term and no scattering. The connected correlation function which gives the fluctuation part is given by  $G(x, y)G(y, x)$  where  $G(x, y)$  is the thermal propagator. We can take the equal-time limit of the usual propagator since  $N(x)$  is the time-independent density. For a uniform gas this gives

$$\begin{aligned} \langle N(x)N(y) \rangle &\equiv \text{Tr}[\rho N(x)N(y)] \\ &= G(x, y)G(y, x) \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} (1 \pm n_k)n_p \end{aligned} \quad (18)$$

where the  $\pm$  signs apply to bosonic and fermionic atoms, respectively. In the high temperature limit, when the occupation numbers  $n_k$  are small,  $n_k \ll 1$ , we get

$$\begin{aligned} \langle N(x)N(y) \rangle &\approx \delta^{(3)}(x-y) \int \frac{d^3p}{(2\pi)^3} n_p \\ &\approx \langle N \rangle \delta^{(3)}(x-y) \end{aligned} \quad (19)$$

where  $\langle N \rangle$  is the average density, taken to be independent of  $x$ . Thus we get, for the scattering from atoms or molecules in a volume  $V$ ,

$$d\sigma = \langle N \rangle V d\sigma_{atom} \quad (20)$$

At this point we might ask ourselves: Do we really need to talk of propagators and quantum field theory to get the result (19). After all, distributing atoms randomly in a volume  $V$ , we should be able to calculate the fluctuations without so much of field theory. And that is indeed true, so here is a poor man's way of getting (19). Divide the volume  $V$  into  $M$  small cells, each of volume  $v$ , so that  $V = Mv$ . We can eventually take  $M \rightarrow \infty$ ,  $v \rightarrow 0$ , keeping  $V$  fixed and finite. Let us label the cells  $1, 2, \dots, M$ . We distribute  $N$  molecules such that we have  $n_k$  molecules in the  $k$ -th cell. The number of ways of doing this is given by the multinomial formula

$$P(\{n_k\}) = \frac{N!}{\prod_k n_k!} \equiv \exp(S(\{n_k\})) \quad (21)$$

Choosing equal *a priori* probability for each cell, this being the hallmark of a truly random arrangement, we can look for the configuration  $\{\bar{n}_1, \bar{n}_2, \dots, \bar{n}_M\}$  which has the maximum probability. This is given by maximizing  $S(\{n_k\})$  subject to  $\sum_k n_k = N$ . Using Stirling's formula  $\log n! \approx n \log n - n$ , which applies for large  $n$ , we can expand  $S(\{n_k\}) + \lambda(\sum_k n_k - N)$  as

$$S(\{n_k\}) = \log N! - \sum_k (n_k \log n_k - n_k) + \lambda \left( \sum_k n_k - N \right)$$

$$= S(\{\bar{n}_k\}) - \sum_k (\log \bar{n}_k - \lambda) \delta n_k - \frac{1}{2} \sum_k \frac{\delta n_k \delta n_k}{\bar{n}_k} + \dots \quad (22)$$

Clearly the maximum occurs when  $\log \bar{n}_k = \lambda$ , or  $\bar{n}_k = e^\lambda \equiv \bar{N}$ , which is independent of  $k$ . The configuration of maximum probability is thus a distribution of uniform number density. For the probability distribution of fluctuations from this mean density, namely, for  $\delta n_k = n_k - \bar{N}$ , we find

$$P(\delta n_k) = C \exp\left(-\sum_k \frac{\delta n_k \delta n_k}{2\bar{N}}\right) \quad (23)$$

where  $C$  is to be determined by the normalization condition of total probability being unity. This equation shows that the mean square fluctuations in density is given by

$$\langle (n_k - \bar{N})(n_l - \bar{N}) \rangle = \bar{N} \delta_{kl} \quad (24)$$

In the continuum limit with  $M \rightarrow \infty$ ,  $v \rightarrow 0$  and where each cell is labeled by the coordinates of its position,

$$\langle (N(x) - \bar{N})(N(y) - \bar{N}) \rangle = \bar{N} \delta^{(3)}(x - y) \quad (25)$$

The Gaussian distribution (23) is centered at  $\bar{N}$ , so that  $\bar{N} = \langle N \rangle$ , and we have recovered (19). (Our counting did not take account of the identity of particles or the exclusion principle, so the result is applicable only for small values of the average occupation numbers. The full result (18) can be obtained if we include the identity of particles in the combinatorics.)

Even though the calculation we have presented does show the importance of density fluctuations and the emergence of the  $\omega^4$ -dependence, the universality of this behavior is not obvious. So we now present another argument based on the effective action and its symmetries which will lead to the same results in a more general framework.

## 2 Effective action, invariance arguments

For low energy elastic scattering, we can use the effective action for the electromagnetic field. With the interaction term written as in (4), this is given by

$$\mathcal{S}_{eff} = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \int d^4x d^4y F_{\mu\nu}(x) F_{\alpha\beta}(y) \langle M^{\mu\nu}(x) M^{\alpha\beta}(y) \rangle \quad (26)$$

The fact that the second term involves  $F$ 's is related to gauge invariance and current conservation; we used current conservation to arrive at (3). In a uniform medium, we can determine the general properties of the correlation function  $\langle M^{\mu\nu}(x) M^{\alpha\beta}(y) \rangle$

as follows. If we assume isotropy and homogeneity, then the Fourier transform of this correlation function can depend only on Fourier variable  $k_\mu$  and on the four-velocity of the medium  $u_\mu$ . It may thus be decomposed in terms of these vectors and the metric tensor  $\eta_{\mu\nu}$ . For long wave length photons, terms proportional to  $k_\mu$  are subdominant. The leading terms are thus made of  $\eta$ 's and  $u_\mu$ . There are only two tensor structures consistent with the antisymmetry of  $M^{\mu\nu}$ , so that we may write

$$\begin{aligned} \langle M^{\mu\nu}(x)M^{\alpha\beta}(y) \rangle &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left[ \frac{1}{2} f(k^2, k \cdot u) (\eta^{\mu\alpha} \eta^{\nu\beta} - \eta^{\mu\beta} \eta^{\nu\alpha}) \right. \\ &\quad \left. + \frac{1}{2} h(k^2, k \cdot u) (\eta^{\mu\alpha} u^\nu u^\beta - \eta^{\mu\beta} u^\nu u^\alpha - \eta^{\nu\alpha} u^\mu u^\beta + \eta^{\nu\beta} u^\mu u^\alpha) \right] \end{aligned} \quad (27)$$

Here  $f$  and  $h$  are functions of the invariants  $k^2$  and  $k \cdot u$  as indicated. In the rest frame of the medium, with  $u^0 = 1$ ,  $\vec{u} = 0$ , we can now write the effective action as

$$\begin{aligned} \mathcal{S}_{eff} &= \frac{1}{2} \int d^4x d^4y E_i(x) \left[ \delta^{(4)}(x-y) + f(x-y) + h(x-y) \right] E_i(y) \\ &\quad - \frac{1}{2} \int d^4x d^4y B_i(x) \left[ \delta^{(4)}(x-y) + f(x-y) \right] B_i(y) \end{aligned} \quad (28)$$

$f(x-y)$ ,  $h(x-y)$  indicate the Fourier transforms of  $f(k^2, k \cdot u)$  and  $h(k^2, k \cdot u)$ , respectively. This identifies the dielectric constant and magnetic permittivity as

$$\begin{aligned} \epsilon(k) &= 1 + f(k^2, k \cdot u) + h(k^2, k \cdot u) \\ \mu^{-1}(k) &= 1 + f(k^2, k \cdot u) \end{aligned} \quad (29)$$

We can use this to calculate the scattering from, as an example, a small dielectric sphere. This is done by treating  $\frac{1}{2}E(\epsilon - 1)E$  as an interaction term. The amplitude is then trivially obtained as

$$\begin{aligned} \mathcal{A} &= i \int \frac{d^4\tilde{k}}{(2\pi)^4} [\epsilon(\tilde{k}) - 1] e^{i\tilde{k}(x-y)} \omega \omega' \vec{\epsilon} \cdot \vec{\epsilon}' \frac{e^{-ikx + ik'y}}{\sqrt{2\omega V 2\omega' V}} \\ &= i[\epsilon(k) - 1] \omega \omega' \frac{2\pi \delta(\omega - \omega')}{\sqrt{2\omega V 2\omega' V}} \vec{\epsilon} \cdot \vec{\epsilon}' \int_{sphere} d^3x e^{i(\vec{k} - \vec{k}') \cdot \vec{x}} \end{aligned} \quad (30)$$

By direct evaluation we see that

$$\begin{aligned} \int_{sphere} d^3x e^{i\vec{q} \cdot \vec{x}} &= \frac{4\pi}{q^3} (\sin qR - qR \cos qR) \equiv F(q) \\ &\Rightarrow \frac{4\pi R^3}{3}, \quad \text{as } q \rightarrow 0 \end{aligned} \quad (31)$$

We can now square the amplitude and calculate the cross section the usual way. For unpolarized incoming photons, we get

$$d\sigma = \frac{|\epsilon(k) - 1|^2}{32\pi^2} \omega^4 |F(\vec{q})|^2 (1 + \cos^2 \theta) d\Omega \quad (32)$$

where  $\vec{q} = \vec{k} - \vec{k}'$  and  $\theta$  is the scattering angle defined by  $\vec{k} \cdot \vec{k}' = \omega^2 \cos \theta$ . This is the standard Rayleigh result for scattering from a dielectric sphere. The quantity  $(\epsilon - 1)$  is related to the polarizability of the material of the sphere. For a denser medium, the factor  $(\epsilon - 1)$  is replaced by  $3(\epsilon - 1)/(\epsilon + 2)$  in accordance with the Clausius-Mosotti relation or the Lorenz-Lorentz formula. The universality of the  $\omega^4$ -dependence is clear from this analysis; it is ultimately traced to the fact that the effective action involves  $F$ 's at low energies.

Even though we used Lorentz invariance for the first part, it is not so important. The amplitude only required the  $(\epsilon - 1)E^2$  term. The key point is that this should involve  $E_i$ 's rather than the vector potential  $A_i$  because of gauge invariance. In replacing the  $E$ 's by the photon wave functions, since  $E_i \sim \omega A_i$ , we get a factor of  $\omega^2$  in the amplitude and, hence, a factor  $\omega^4$  in the cross section.

Suppose now that there are several scattering centers, each of which may be approximated by a small dielectric sphere. Let  $\vec{a}_n$  denote the center of the  $n$ -th sphere. We then get

$$\begin{aligned} F(q) &= \int d^3x e^{i\vec{q}\cdot\vec{x}} = \sum_n e^{i\vec{q}\cdot\vec{a}_n} \int d^3\xi e^{i\vec{q}\cdot\xi} \\ &= \int d^3x e^{i\vec{q}\cdot\vec{x}} N(x) F_1(q) \end{aligned} \quad (33)$$

where  $F_1(q)$  refers to a single scatterer. This formula leads to the cross section

$$d\sigma = d\sigma_1 \int d^3x d^3y e^{i(\vec{k}-\vec{k}')\cdot(\vec{x}-\vec{y})} \langle N(x)N(y) \rangle \quad (34)$$

This agrees with the previous calculation, formula (15).

### 3 Gauge invariance, the blue sky and the blue ocean

The  $\omega^4$ -dependence can explain the blue color of the sky. The blue part of sunlight, which has higher  $\omega$ , gets scattered more and ends up as diffuse light in the atmosphere. The yellow part of the spectrum is not scattered so much, is retained as more direct sunlight, making the sun appear yellow. In the evenings, or early in the mornings, the sunlight has to traverse a longer distance through the atmosphere and even the yellow gets scattered significantly. Even lower frequencies, such as the red, are what is obtained as direct light, giving us the reddish colors of sunsets and sunrises. This is the general picture, but, of course there are many fine tunings to be done. For the actual atmosphere, we cannot take the density to be absolutely constant, instead, it follows a distribution, thinning out as we go to higher elevations. Other sources of variations in the density of scatterers such as weather conditions are also important.



Figure 3: Earth from space - the “blue marble” [3]

Nevertheless, the crucial ingredients are the  $\omega^4$ -dependence and statistical fluctuations in density, traceable to gauge invariance of the effective action and the very nature of statistical distributions. So here is an “elevated thought” to entertain when looking up at the sky - gauge invariance and density fluctuations!

What can we say about the blue ocean? Why does water, transparent in small quantities, take on such an intense blue color when amassed in large quantities in the ocean, so much so that our planet appears as the “pale blue dot” in the vastness of space? Could it be that the ocean reflects the sky? The deeper more intense blue of

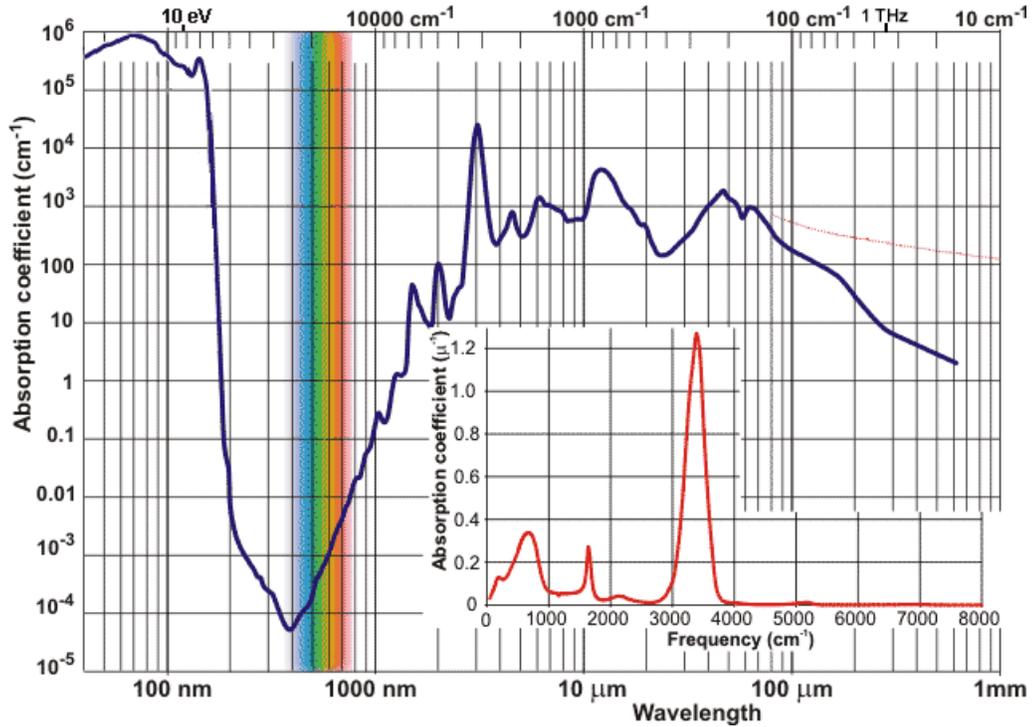


Figure 4: Absorption spectrum of water ( $\text{H}_2\text{O}$ , not  $\text{D}_2\text{O}$ ) [4]

the ocean is an argument to discount this possibility; such an effect, at best, can have only a minor role. Pictures of Earth from space show clearly that the light coming off the ocean is blue, as in the beautiful “blue marble” picture, Fig. 3. But the atmospheric effect must be small as there is no discernible blue haze over the land masses. The blue ocean must be understood in terms of light absorption properties of water.

Despite its abundance, despite being nearly three-quarters of our bodily constitution, water is such a complex liquid that its properties are still being investigated [5, 6]. The absorption spectrum for wavelengths near the visible range is as shown in Fig. 4. A more detailed version of the absorption in the visible region is shown in Fig. 5. Absorption is high in the near infrared and for radio waves (even for longer wave lengths than what is shown in the figure, which is one reason we cannot use radar for submarine detection under water), becoming smaller in the visible region. Of the visible wave lengths, the red end of the spectrum is more strongly absorbed compared to the blue end. As a result, while absorption is negligible for light propagating for small distances in water, making it seem transparent, over longer distances the light which survives is mostly blue. However, the light we see form the ocean is not transmitted light, so even though the blue wave lengths survive, they have to get

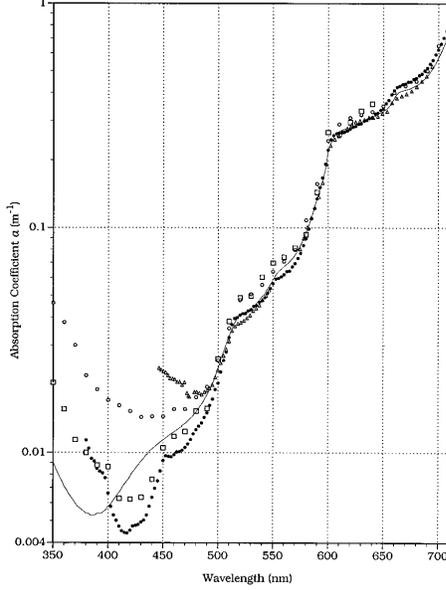


Figure 5: Absorption spectrum of water in and near the visible wave lengths [7]

out of the water to come to our eyes. This is of course a process of multiple scattering. So at the end of the day, we do need scattering to explain the blue of the ocean, although it is not directly for the same reason as for the blue of the sky.

## 4 Remarks

The formula for scattering, in terms of the correlation function for densities, can also explain the phenomenon of critical opalescence. Near a liquid-vapor critical point, the density fluctuations become very large, and hence the scattering cross section increases, leading to an opalescent appearance for the liquid-vapor mixture. The standard method to analyze this effect is to relate it to the compressibility of the material and use data on compressibility. But one can also think of this more directly in terms of the behavior of the correlation function. At the critical point, this is expected to be of the form

$$\langle N(x) N(y) \rangle = \frac{C}{|x-y|^\eta} \sim \int \frac{d^3q}{(2\pi)^3} \frac{e^{iq \cdot (x-y)}}{|q|^{3-\eta}} \quad (35)$$

where  $C$  is a constant. The exponent  $\eta$  characterizes the behavior of the correlation function. We see from (15) and the Fourier version here, that we do get a divergence for low energy scattering as  $|q| = |\vec{k} - \vec{k}'| \sim \omega$  goes to zero.

For water, the critical point, where the distinction between the liquid and vapor phases disappears, occurs at a temperature of around 647 K and a pressure of 218 atmospheres. Relating critical opalescence to Rayleigh scattering is something which goes back to Einstein himself, more than a century ago. Yet the understanding of correlation functions at criticality has emerged only over the course of the last few decades, the most striking aspect of which is the concept of universality. Critical behavior of physical systems fall into universality classes, where the fine details of the system and various interactions are not important, but certain universal behavior emerging out of the cumulative effect of the interactions is all that matters. The critical behavior for water is in the same universality class as the three-dimensional Ising model. Thus we can determine the nature of the correlation function  $\lambda N(x) N(y)$  at criticality by looking at the correlation function for the spin field (or the self interacting scalar field in three dimensions). This is the classic case which was analyzed in the framework of the renormalization group and the  $\epsilon$ -expansion by Wilson and Fischer. Their result can be expressed as  $\eta = \frac{1}{2} + (1/108)$ , to the lowest order in the calculation [8]. Since then there have been many other approaches, numerical as well as analytical. The most recent attempts include the conformal bootstrap which is based on the algebra of the relevant operators of the theory at criticality, a development of the algebraic approach to quantum field theory. This method allows for the calculation of the critical exponents to a high degree of accuracy, giving, in the present case,  $\eta = 0.518151(6)$  [9]. It is quite striking, and in a way delightfully so, that something like the algebra of field operators in a quantum field theory can have a direct impact on a phenomenon like critical opalescence or Rayleigh scattering.

It is also amusing to see how these formulae apply to some unusual situations. Consider, for example, scattering from a medium made of small globules of superconducting material. In a superconductor, the transverse photons have a mass; this will dominate for low energies or long wave lengths. Treating this mass term as the interaction term, the amplitude for scattering is of the form

$$\mathcal{A} = i\mu^2 \int d^4x \epsilon \cdot \epsilon' \frac{e^{-ikx+ik'x}}{\sqrt{2\omega V 2\omega' V}} \quad (36)$$

Notice the absence of the  $\omega\omega'$  in the numerator compared to (30); the latter was due to  $F$ 's rather than  $A$ 's in the action. Also, the polarization tensors, it must be kept in mind, are transverse to the corresponding wave vectors. The rest of the analysis is very similar to what we have already done and will lead to a formula like (32),

$$d\sigma = \frac{\mu^4}{32\pi^2} |F(\vec{q})|^2 (1 + \cos^2 \theta) d\Omega \quad (37)$$

If the atmosphere of some alien planet is made of a distribution of globules of superconducting material, the residents would not have the benefit of blue skies and orange-red sunsets.

Letting the mind wander more, consider now a stack of discs, each of which is in a quantum state displaying the quantum Hall effect. For a quantum Hall droplet in two dimensions, the expectation value of the current is given by

$$\langle J_i(x) \rangle = \frac{e^2 \nu}{2\pi} \epsilon_{ijk} \hat{n}_k F_{0j}(x) \quad (38)$$

where  $\hat{n}_k$  denotes the vector normal to the surface of the droplet and  $\nu$  is the filling fraction. This equation shows that

$$\langle J_i(x) J_j(y) \rangle = \frac{e^2 \nu}{2\pi} \epsilon_{ijk} \hat{n}_k \partial_0 \delta^{(4)}(x - y) \quad (39)$$

The scattering amplitude is then obtained as

$$\begin{aligned} \mathcal{A} &= i \frac{e^2 \nu}{2\pi} \int d^4x \omega' \frac{(\vec{\epsilon} \times \vec{\epsilon}') \cdot \hat{n}}{\sqrt{2\omega V 2\omega' V}} e^{-i(k-k')x} \\ &= i \frac{e^2 \nu}{2\pi} \omega' \frac{(\vec{\epsilon} \times \vec{\epsilon}') \cdot \hat{n}}{\sqrt{2\omega V 2\omega' V}} F(\vec{q}) 2\pi(\delta(\omega - \omega')) \end{aligned} \quad (40)$$

This leads to the cross section

$$d\sigma = \left( \frac{e^2 \nu}{2\pi} \right)^2 \frac{\omega^2}{16\pi^2} |F(\vec{q})|^2 |(\vec{\epsilon} \times \vec{\epsilon}') \cdot \hat{n}|^2 \quad (41)$$

Carrying out the polarization sums and averaging, we get

$$d\sigma = \left( \frac{e^2 \nu}{2\pi} \right)^2 \frac{\omega^2}{32\pi^2} |F(\vec{q})|^2 \left[ \frac{(\vec{k} \cdot \hat{n})^2 \omega^2 + (\vec{k}' \cdot \hat{n})^2 \omega'^2 + [(\vec{k} \times \vec{k}') \cdot \hat{n}]^2}{\omega^4} \right] \quad (42)$$

The cross section has a milder, namely  $\omega^2$ , dependence on the frequency.  $F(\vec{q})$ , as usual, will carry information about the distribution of the quantum Hall droplets. Thus, if we consider quantum Hall droplets in the  $(x, y)$ -plane, stacked along the  $z$ -axis at positions  $z_n$ , we can write

$$F(\vec{q}) = \int_{disc} d^2x e^{iq_1 x + iq_2 y} \sum_n e^{iq_3 z_n} \quad (43)$$

It is clear that the effects of different geometrical arrangements can be evaluated easily from the general formula for  $F(\vec{q})$ .

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