

QFT & Scattering amplitudes

6/30/13

L1

at Open Univ., Tokyo

Plan

1. Introduction.

2. Real scalar fields.

3. Interacting fields

4. Decay rates & cross sections.

5. Electromagnetic fields.

6. List of S-matrix functionals.

à la Nair.

formal aspects.

Things not to cover.

1. Non-relativistic (Schrödinger) fields

2. Dirac fields

3. Non abelian gauge fields

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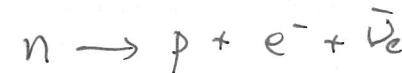
QED, QCD

Standard model.

1. Introduction.

Why QFT?

QM of many particles

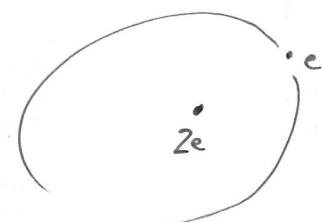


p-decay



pair annihilation

QM of fields



H. like atom

Uehling potential

$$V(r) \approx -Z e^2 \left[\frac{1}{r} + (\alpha) \delta^{(1)}(r) \right]$$

$$\alpha = \frac{1}{137}$$

on Vacuum polarization effect.

Quantization of (\vec{F}, \vec{B})

Bohr & Rosenfeld

QFT

two approaches.

1. QM of many particles \rightarrow Construction of field operators2. QM of fields \rightarrow Field quantization

(Canonical quantization)

Particles

fields

Higgs, π^0
pseudo scalar

spin-0 bosons

real scalar field

 π^\pm, k^\pm mesons

Charged spin-0 bosons

complex scalar field

spin- $1/2$ fermions

spinor fields

spin-1 massless boson

real vector fields

How to quantize classical theory

$$S = \int dt L(x) = \int dt dx' \mathcal{L}.$$

Rules of quantization

1. States of the physical system \Leftrightarrow vectors (rays) in \mathcal{H} .

2. Observables \Leftrightarrow linear Hermitian operators on \mathcal{H} .

3. For any operators \hat{A} , $\delta\hat{A} = \frac{[\hat{A}, \Theta]}{i\hbar}$

Schwinger's

where

action principle

$$\delta S = \Theta(t_f) - \Theta(t_i) + \int dt (\text{Eqn. of motion}) \delta x.$$

Θ : Canonical 1. form



Canonical CR's

1-particle problem

$$L = \frac{m}{2} \dot{x}^2 - V(x) \quad \tilde{x}(t) = x^i(t) \quad i=1, 2, 3$$

$$\delta S = \int dt \left(m \dot{x}^i \delta x^i - \underbrace{\frac{\partial V}{\partial x^i} \delta x^i}_{\text{Newton's eqn. of motion}} \right)$$

$$\frac{d}{dt} (m \dot{x}^i \delta x^i) - m \ddot{x}^i \delta x^i$$

$$= [m \dot{x}^i \delta x^i]_{t_i}^{t_f} - \int dt \left(m \ddot{x}^i + \frac{\partial V}{\partial x^i} \right) \delta x^i \quad \underbrace{= 0}_{\text{Newton's eqn. of motion}}$$

$$\textcircled{1} : m \dot{x}^i \delta x^i = p^i \delta x^i$$

$$\delta \hat{A} = \frac{[\hat{A}, p^i \delta x^i]}{i\hbar}$$

arise when

$$\delta x^i(t_i) \neq 0$$

$$\delta x^i(t_f) \neq 0$$

Fixing the initial & final points

$$\delta x^i = 0 \text{ at } t_i, t_f$$

The trajectory obeys $\delta S = 0$

Newton's eqn. of motion

\ddot{x}^i is given

freedom to choose x^i, \dot{x}^i

$$\underline{\text{Set}} \quad \delta x^i = a^i, \delta p^i = 0$$

$$\hat{A} = \hat{x}^i$$

$$\Rightarrow a^i = \frac{[x^i, p^j] a^j}{i\hbar} \Rightarrow [x^i, p^j] = i\hbar \delta^{ij}$$

$$\hat{A} = \hat{p}^i$$

$$\Rightarrow 0 = \frac{[p^i, p^j] a^j}{i\hbar} \Rightarrow [p^i, p^j] = 0$$

② commutation relation changing the role of $p \leftrightarrow x$

$$\Rightarrow [x^i, x^j] = 0$$

Action principle \rightarrow Heisenberg algebra.

2.

Real scalar fields.

$$S = \int_{t_i}^{t_f} dt \int d^3x \mathcal{L}(\phi, \partial_\mu \phi)$$

$$x^\mu \quad (\mu = 0, 1, 2, 3)$$

$$\partial_\mu \phi = \frac{\partial}{\partial x^\mu} \phi$$

$$(\partial \phi)^2 = (\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2 - (\partial_3 \phi)^2$$

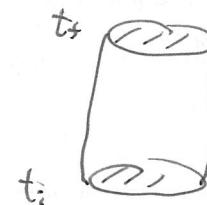
$$\delta S = \int dt d^3x \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi \right)$$

$$g_{\mu\nu} = (+ - - -)$$



$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta_\mu \delta \phi$$

$$= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \right] - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi$$



$$= \int dt d^3x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \delta \phi + \int dt d^3x \left[\partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \delta \phi - \partial_1 \frac{\partial \mathcal{L}}{\partial (\partial_1 \phi)} \delta \phi \right]$$

$$\underbrace{\left[\int d^3x \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \delta \phi \right]_{t_i}^{t_f} - \cancel{\int dt \int d^3x \partial_0 \left(\frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \delta \phi \right)}}$$

$$(H)(t) = \int d^3x \frac{\partial \mathcal{L}}{\partial (\partial_0 \phi)} \delta \phi$$

: Canonical 1. form.

$\delta \phi = 0$ at spatial boundary
but $\delta \phi \neq 0$ at t_i, t_f

Free scalar field.

$$\mathcal{L} = \frac{1}{2} (\partial\phi)^2 - \frac{m^2}{2} \phi^2$$

$$\mathbb{H} = \int d^3x \ \pi(x) \dot{\phi} \quad \pi = \partial_0 \phi = \dot{\phi}$$

$$; \delta \hat{A} = [\hat{A}, \mathbb{H}] \quad h=1 \quad (\text{from here on})$$

$$\delta\phi = \epsilon(x), \quad \delta\pi = 0 \quad \hat{A} = \phi(y)$$

$$; \epsilon(y) = [\phi(y), \int d^3x \pi(x) \epsilon(x)]$$

$$= \int d^3x ; \delta^{(3)}(x-y) \epsilon(x)$$

$$\Rightarrow [\phi(y), \pi(x)] = ; \delta^{(3)}(x-y)$$

$$\hat{A} = \pi(y) \quad \Rightarrow \quad [\pi(y), \pi(x)] = 0$$

$$[\phi(x), \phi(y)] = 0$$

CR's hold for free & interacting theories
as long as the interacting term does not
involve time-derivative of the fields $\mathcal{L}_{\text{int}}(\phi, \dot{\phi})$

Equal time Commutation Rules

Field equation

$$-k^2\phi - \partial_m \partial^m \phi = 0$$

$$(\square + m^2)\phi = 0$$

$$\partial_m \partial^m \phi = \frac{\partial^2}{\partial t^2} \phi - \nabla^2 \phi = \square \phi$$

Klein-Gordon eqn.

Canonical CR's
KG eqn.

Quantities of interest

$$\left. \begin{array}{l} H = \int d^3x \frac{1}{2} (\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2) \\ P_i = + \int d^3x \dot{\phi} \partial_i \phi \end{array} \right\}$$

→ One can obtain many-particle QM system.

Canonical transformation
Response of the system to the change of time.

Change of space coord.

$x^i \rightarrow x_i + \xi^i$
Energy-momentum tensor
 $T_{ij}^{\mu\nu}$

field quantization.

Construction of field operators

KG eqn.

$$(D + m^2) \phi = 0$$

$$\partial^2 \phi - \omega^2 \phi + m^2 \phi = 0$$

$$\phi = A e^{-i\omega t + i\vec{k} \cdot \vec{x}}$$

$$\omega = \sqrt{\vec{k}^2 + m^2}$$

mode expansion

$$\phi(x) = \sum_k \left(a_k u_k(x) + a_k^* u_k^*(x) \right)$$

$$u_k(x) = \frac{e^{-i\omega_k t + i\vec{k} \cdot \vec{x}}}{\sqrt{2\omega_k V}}$$

$$\omega_k = \sqrt{\vec{k}^2 + m^2}$$

$$V = L^3$$



$$k = \frac{2\pi}{L} n$$

$$e^{ik_1 L} = 1$$

periodic b.c.

$$H = \frac{1}{2} \sum_k \omega_k (a_k a_k^* + a_k^* a_k) = \sum_k \omega_k (a_k^* a_k + \frac{1}{2})$$

CR's

$$\rightarrow [a_k, a_\ell] = 0$$

$$[a_k^*, a_\ell^*] = 0$$

$$[a_k, a_\ell^*] = \delta_{k,\ell}$$

$$P_i = \sum_k k_i (a_k^* a_k + \frac{1}{2}) \Rightarrow \sum_k k_i a_k^* a_k$$

$$P|0\rangle = H|0\rangle = 0$$

Lorentz inv.

Lorentz invariance of the ground state $|0\rangle$

$$\mathcal{P}|0\rangle = 0$$

$$H|0\rangle = 0, \quad a_k|0\rangle = 0$$

$a_k^\dagger|0\rangle$: 1-particle state of momentum k_i
energy ω_k .

If we do not have Lorentz invariance?

Curved space

"Vacuum energy" is physically observable.

e.g., Hawking radiation. (black body radiation of black holes due to quantum effect)
Casimir effect (forces b/w conducting plates in vacuum)

Propagator

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle$$

$$\text{use page 11} \quad = \int \frac{d^4 k}{(2\pi)^4} e^{ik(k-y)} \frac{i}{k^2 - m^2 + i\epsilon}$$

$$\equiv G(x, y)$$

$$(\square + m^2 - i\epsilon) G(x, y) = -i \delta^{(4)}(x-y)$$

T: time ordering

$$T \phi(x) \phi(y) = \begin{cases} \phi(x) \phi(y) & \text{when } x^0 > y^0 \\ \phi(y) \phi(x) & \text{when } y^0 > x^0 \end{cases}$$

$$\left(\begin{array}{l} k^2 = k_0^2 - \vec{k}^2 \\ k_0 = \omega_k \end{array} \right)$$

Green's function
in the mathematical sense.

$$\left[G(x, y) \leftrightarrow \frac{i}{\square + m^2 - i\epsilon} \right]$$

For $x^0 > y^0$

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \sum_{k,\ell} \langle 0 | (a_k u_k(x) + a_k^\dagger u_k(x)) (a_\ell u_\ell(y) + a_\ell^\dagger u_\ell(y)) | 0 \rangle$$

$$= \sum_k u_k(x) u_k^*(y)$$

↑

$$\langle 0 | a_k a_\ell^\dagger | 0 \rangle = \langle 0 | (a_k a_\ell^\dagger - a_\ell^\dagger a_k) | 0 \rangle$$

$$= \delta_{k\ell} \langle 0 | 0 \rangle = \delta_{k\ell}$$

For $x^0 < y^0$

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle = \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\omega_k(x^0-y^0)-ik(\vec{x}-\vec{y})}}{2\omega_k} \vec{k} = \frac{2\pi}{L} \vec{n}$$

~~Wavy lines~~

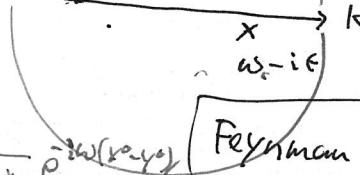
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Similar for $x^0 < y^0$

$$\oint \frac{e^{-ik(x^0-y^0)}}{k^2 - \omega^2} \quad \boxed{15}$$

Feynman propagator $= \frac{2\pi i}{2(-\omega_k)} \frac{e^{i\omega_k(x^0-y^0)}}{2(-\omega_k)}$

$$= -2\pi i \frac{e^{i\omega_k(x^0-y^0)}}{2\omega_k}$$



Feynman contour.

See / any textbook.

$$\langle 0 | T \phi(x) \phi(y) | 0 \rangle$$

$$= \int \frac{d^3 k}{(2\pi)^3} \frac{dk}{2\pi} \frac{i e^{-ik(x^0-y^0)+iE(\vec{k}-\vec{y})}}{\vec{k}^2 - m^2 + i\epsilon}$$

~~Propagator~~ Why this is called propagator.

1-particle QM

$$\omega_k = \sqrt{\vec{p}^2 + m^2}$$

$|\vec{y}\rangle$: particle at position \vec{y} at time y^0

The Probability amplitude to detect a particle at \vec{x} at time x^0

$$(\text{Probability}) = \langle \vec{x} | e^{-iH(x^0-y^0)} |\vec{y}\rangle$$

$$\langle \vec{x} | k \rangle = \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}$$

$$\int |k\rangle \langle k| \frac{d^3 k}{2\omega_k} = 1$$

$$= \int \underbrace{\langle \vec{x} | k \rangle}_{e^{i\vec{k}\cdot\vec{x}} / (2\pi)^{3/2}} \underbrace{\langle k | \vec{y} \rangle}_{e^{-i\vec{k}\cdot\vec{y}} / (2\pi)^{3/2}} e^{-i\omega_k(x^0-y^0)} \frac{d^3 k}{2\omega_k}$$

= propagator

$$H \rightarrow i\hbar \frac{\partial}{\partial t}$$

Prob. amplitude to detect a particle at (x^0, \vec{x}) if it was introduced at (y^0, \vec{y}) when $x^0 > y^0$.

Generalization

L12

N-point function / correlation function

$$G(x_1, \dots, x_N) = \langle 0 | T \phi(x_1) \phi(x_2) \dots \phi(x_N) | 0 \rangle$$

A succinct way to describe these

$$\sum [J] = \sum \frac{1}{N!} \int d^4 x_1 \dots d^4 x_N G(x_1, \dots, x_N) J(x_1) \dots J(x_N)$$

$$= \langle 0 | T e^{\int J \phi} | 0 \rangle$$

$J(x)$: arbitrary function of x

source function (Not operator)

C-number function

OR

$$G(x_1, x_2, \dots, x_N) = \frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_N)} \sum [J]$$

$$\bigg|_{J=0}$$

Derive Equation for $\sum [J]$

In general functional derivatives are defined as

$$\delta I[\phi] = \int d^4 x \left(\frac{\delta I}{\delta \phi} \right)$$

generalization of $(\Box + m^2) G(x, y) = -i \delta^{(4)}(x-y)$

$$I = \int d^4 x \left[\frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \right]$$

where $\phi = \phi^{(0)}$ is the classical field solution to the equations of motion (2.3). This is a key step in deriving the Feynman rules for the theory.

Functional derivatives

$I[\phi] : \{ \text{set of real functions on } \mathbb{X}^n \} \rightarrow \mathbb{R}$

With finiteness condition

$$\int d^4x \phi^2 < \infty, \quad \int d^4x |\phi|^2 < \infty$$

$\frac{\delta I[\phi]}{\delta \phi}$ is defined by

$$\boxed{\delta I = \int d^4x \left[\frac{\delta I}{\delta \phi} \right] \delta \phi}$$

e.g.

$$\frac{\delta \phi(x)}{\delta \phi(y)} = \delta^{(4)}(x-y)$$

$$\frac{\delta}{\delta \phi(x)} \int d^4y \phi^2(y) = 2\phi(x)$$

$$\frac{\delta}{\delta \phi(x)} \int d^4x (\partial \phi)^2 = 2(-\square \phi(x))$$

$$\leftarrow \delta \int d^4y (\partial \phi)^2 = \int d^4y \underbrace{2 \partial \phi \delta(\partial \phi)}_{\partial(\partial \phi \delta \phi)} - \square \phi \delta \phi$$

Eqs. of motion

$$\frac{\delta S}{\delta \phi} = 0. \quad (\text{with finiteness condition})$$

$$\frac{\delta \Sigma[J]}{\delta J(x)} = \sum_N \frac{1}{N!} \int_Y J(y_1) \dots J(y_N) \langle 0 | T \phi(x) \phi(y_1) \dots \phi(y_N) | 0 \rangle \quad \leftarrow \text{from the definition of } \frac{\delta S}{\delta J} \quad [14]$$

OR.

$$\frac{\delta}{\delta J(x)} T e^{\int J \phi} = e^{\int_{t_{N-1}}^{t_N} dy J \phi} e^{\int_{t_1}^{t_2} J \phi} \underbrace{\phi(x) e^{\int_{t_2}^{x^0} J \phi}}_{T e^{\int_{x^0}^\infty J \phi}} \dots e^{\int_{t_{N-1}}^{t_N} J \phi} = T \phi(x) e^{\int J \phi}$$

Calculate $\square_x \left(\frac{\delta \Sigma}{\delta J} \right)$

Taking the time-derivative

$$\begin{aligned} & \frac{\partial}{\partial x^0} \langle 0 | T \phi(x) e^{\int J \phi} | 0 \rangle \\ &= \langle 0 | T e^{\int_{x^0}^\infty J \phi} \frac{\partial \phi}{\partial x^0} T e^{\int_{-\infty}^{x^0} J \phi} | 0 \rangle \\ &+ \langle 0 | T e^{\int_{x^0}^\infty J \phi} \left\{ \left[- \int d^3y J(x^0, \vec{y}) \phi(x^0, \vec{y}) \right] \phi(x^0, \vec{x}) \right\} T e^{\int_{-\infty}^{x^0} J \phi} | 0 \rangle \\ &+ \phi(x^0, \vec{x}) \int d^3y J(x^0, \vec{y}) \phi(x^0, \vec{y}) \\ &= \int d^3y J(x^0, \vec{y}) \langle 0 | T e^{\int_{x^0}^\infty J \phi} [\phi(x^0, \vec{x}), \phi(x^0, \vec{y})] T e^{\int_{-\infty}^{x^0} J \phi} | 0 \rangle \end{aligned}$$

$$\frac{\partial^2}{\partial x^0 \partial x^1} = \frac{\partial}{\partial x^0} \langle 0 | T \dot{\phi}(x) e^{\int J \phi} | 0 \rangle$$

$$\begin{aligned} &= \langle 0 | T \ddot{\phi}(x) e^{\int J \phi} | 0 \rangle \\ &+ \int d^3y J(x^0, \vec{y}) \langle 0 | T e^{\int_{x^0}^\infty J \phi} \underbrace{[\dot{\phi}(x), \phi(\vec{y})]}_{-i \delta^{(3)}(\vec{x} - \vec{y})} T e^{\int_{-\infty}^{x^0} J \phi} | 0 \rangle \end{aligned}$$

" CR's.

$$= -i J(x) \Sigma[J]$$

The space-derivatives ∇_x go through the time-ordering symbol

$$(-\nabla_x^2 + m^2) \left(\frac{\delta \Sigma}{\delta J} \right) = \langle 0 | T (-\nabla_x^2 + m^2) \phi e^{\int J \phi} | 0 \rangle$$

$$\Rightarrow (\Box_x + m^2) \left(\frac{\delta \Sigma}{\delta J} \right) = \underbrace{\langle 0 | T [(\Box_x + m^2) \phi] e^{\int J \phi} | 0 \rangle}_{\text{"0 eqn."}} - i J(x) \Sigma[J]$$

$$\stackrel{K.F.}{=} -4 \phi^3 \quad \text{for } \lambda \phi^4 \text{ theory}$$

Equation for Σ generating functional for correlators

$$\boxed{(\Box_x + m^2) \frac{\delta \Sigma[J]}{\delta J(x)} = -i J(x) \Sigma[J]}$$

free theory

Solution

$$\Sigma_0[J] = N e^{\frac{i}{2} \int d^4 k d^4 y J(x) G(x, y) J(y)}$$

$$N=1 \Leftarrow \Sigma[0] = 1$$

$$\begin{aligned} & \langle 0 | T \phi_1 \dots \phi_4 | 0 \rangle \\ &= \frac{\delta^4 \Sigma}{\delta J(k_1) \dots \delta J(k_4)} \Big|_{J=0} \\ &= G(x_1, x_2) G(x_3, x_4) \\ &+ G(x_1, x_3) G(x_2, x_4) \\ &+ G(x_1, x_4) G(x_2, x_3) \end{aligned}$$

symmetrized "boson"

Not true for interacting theories.

(Next)

check Using $(\Box_x + m^2) G(x, y) = -i \delta^{(4)}(x-y)$

$$\frac{\delta}{\delta J} \left[e^{\frac{i}{2} \int_{x,y} J \phi J} \right] = e^{\frac{i}{2} \int J \phi J} \int dz G(x, z) J(z)$$

$$(\Box_x + m^2) \left(\frac{\delta \Sigma}{\delta J} \right) = e^{\frac{i}{2} \int J \phi J} \int_z \underbrace{(\Box_x + m^2) G(x, z) J(z)}_{-i \delta^{(4)}(x-z)} = e^{\frac{i}{2} \int J \phi J} (-i J(x)) = -i J(x) \Sigma[J]$$

$$\boxed{\Sigma_0[J] = e^{\frac{i}{2} \int J G \phi J} = e^{\frac{i}{2} \int_{x,y} J(x) G(x, y) J(y)}}$$

- G Feynman propagator
- b.c. for inverting $\Box + m^2$

Free theory

3. Interacting fields.

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$$S \rightarrow S = S_0 + S_{\text{int}}$$

Free theory This leads to nonlinear terms
in equations of motion

Interacting Scalar field

$$S = \frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + S_{\text{int}}$$

Eqn of motion $(\Box + m^2) \phi(x) - \frac{\delta S_{\text{int}}}{\delta \phi(x)} = 0$

$$\mathcal{L} = \frac{1}{2} ((\partial\phi)^2 - m^2 \phi^2) - \lambda \phi^4$$

\mathcal{L}_{int} ϕ^4 -theory

$$\Rightarrow (\Box + m^2) \phi(x) + 4\lambda \phi^3(x) = 0$$

equal-time

$$[\phi(x^0, \vec{x}), \phi(x^0, \vec{y})] = 0$$

Canonical
commutation
rules

$$[\phi(x^0, \vec{x}), \pi(x^0, \vec{x})] = i \delta^{(3)}(\vec{x} - \vec{y})$$

$$[\pi(x^0, \vec{x}), \pi(x^0, \vec{y})] = 0$$

Equation for $Z[J] = \langle 0 | T e^{\int x J \phi} | 0 \rangle$

$$(D_x + m^2) \frac{\delta Z[J]}{\delta J(x)} = \underbrace{\langle 0 | T (D_x + m^2) \phi}_{\sim 4\lambda \phi^3} e^{\int x J \phi} | 0 \rangle - i J(x) Z[J]$$

$$\begin{aligned} \frac{\delta Z[J]}{\delta J(x)} &= \langle 0 | T \phi e^{\int x J \phi} | 0 \rangle \\ \frac{\delta^2 Z}{\delta J(x) \delta J(y)} &= \langle 0 | T \phi^2(x) e^{\int x J \phi} | 0 \rangle \end{aligned} \quad \Rightarrow \quad \boxed{(D_x + m^2) \frac{\delta Z}{\delta J(x)} + 4\lambda \frac{\delta^3 Z}{\delta J^3(x)} = -i J(x) Z[J]}$$

$$\frac{\delta^3 Z[J]}{\delta J(x)^3} = \langle 0 | T \phi^3(x) e^{\int x J \phi} | 0 \rangle \quad Z_0 = e^{\frac{i}{2} \int J \phi J} \quad \text{interacting theory.}$$

Solution $Z[J] = e^{-i\lambda \int \frac{\delta^4}{\delta J^4(z)} d^4 z} Z_0[J]$ $\int_z \frac{\delta^4}{\delta J^4}$

check $(D_x + m^2) \frac{\delta Z}{\delta J} = e^{-i\lambda \int \frac{\delta^4}{\delta J^4}} \underbrace{(D_x + m^2) Z_0[J]}_{-i J(x) Z_0[J]} = -i e^{-i\lambda \int \frac{\delta^4}{\delta J^4}} (J(x) Z_0[J])$

do not commute.

$$\boxed{e^A B = (\underbrace{e^A B e^{-A}}_{B + [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \dots} e^A)}$$

$$= -i e^{-i\lambda \int \frac{\delta^4}{\delta J^4}} J(x) e^{i\lambda \int \frac{\delta^4}{\delta J^4}} e^{-i\lambda \int \frac{\delta^4}{\delta J^4}} Z[J]$$

$$J(x) + (-i\lambda) \left[\int \frac{\delta^4}{\delta J^4}, J(x) \right] + \frac{(-i\lambda)^2}{2!} \left[\int \frac{\delta^4}{\delta J^4}, \left[\int \frac{\delta^4}{\delta J^4}, J(x) \right] \right] + \dots$$

$$\int dz \left[\frac{\delta^4}{\delta J(z)}, J(x) \right]$$

$$[AB, C] = [A, C]B + A[B, C]$$

$$= 4 \int_z \frac{\delta^3}{\delta J^3} \left[\frac{\delta}{\delta J(z)}, J(x) \right]$$

$$[P^4, x] = 4P^3(-i)$$

$$= 4 \int_z \frac{\delta^3}{\delta J^3} \delta(z-x) = 4 \frac{\delta^3}{\delta J^3(x)}$$

$$P^3 [P, x] + P^2 [P, x] P + \dots$$

$$\left[\int_z \frac{\delta^4}{\delta J^4}, \left[\int_y \frac{\delta^4}{\delta J^4}, J(x) \right] \right] = \int_z \left[\frac{\delta^4}{\delta J^4(z)}, 4 \frac{\delta^3}{\delta J^3(x)} \right] = 0$$

derivatives commute

check

$$\left[\frac{\delta^4}{\delta J^4}, J(x) \right] 24 = \frac{\delta^4}{\delta J^4} (J(x) 24) - J(x) \frac{\delta^4}{\delta J^4} 24$$

$$\begin{aligned} \left[\frac{\delta}{\delta J(y)}, J(x) \right] 24 &= \frac{\delta}{\delta J(y)} J(x) 24 - J(x) \frac{\delta^4}{\delta J^4(y)} \\ &= \delta^{(4)}(x-y) 24 + J(x) \cancel{\frac{\delta^4}{\delta J(y)}} - J(x) \cancel{\frac{\delta^4}{\delta J(y)}} = \delta^{(4)}(x-y) 24 \end{aligned}$$

$$\left[\frac{\delta^2}{\delta J(x)}, J(x) \right] 24 = \underbrace{\frac{\delta^3}{\delta J^3(y)} J(x) 24}_{-} - J(x) \cancel{\frac{\delta^2}{\delta J^2(y)}} 24$$

$$\frac{\delta}{\delta J(y)} \left(\delta^{(4)}(x-y) 24 + J(x) \frac{\delta^4}{\delta J^4(y)} \right)$$

$$= \cancel{J(x)} \delta^{(4)}(x-y) \frac{\delta^4}{\delta J^4(y)} + \delta^{(4)}(x-y) \frac{\delta^4}{\delta J^4(y)} + J(x) \cancel{\frac{\delta^2}{\delta J^2(y)}} 24$$

$$= 2 \delta^{(4)}(x-y) \frac{\delta^4}{\delta J^4(x)}$$

$$\left[\frac{\delta^n}{\delta J^n(y)}, J(x) \right] = n \delta^{(n)}(x-y) \frac{\delta^{n-1}}{\delta J^{n-1}(y)}$$

by induction

skip
補足

$$\Sigma[J] = e^{-i\lambda \int_{\mathbb{R}^4} \frac{\delta^4}{\delta J^4(z)} e^{\frac{i}{2} \int_{x,y} J(x) G(x,y) J(y)}}$$

$$\begin{aligned} (\text{Propagator}) &= \tilde{G}(x,y) := \left. \frac{\delta^2 \Sigma[J]}{\delta J(x) \delta J(y)} \right|_{J=0} \\ &= \frac{\delta}{\delta J(x)} e^{-i\lambda \int_{\mathbb{R}^4} \frac{\delta^4}{\delta J^4}} e^{\frac{i}{2} \int_{x,z_1} J(x) G(x,z_1) J(z_1)} \\ &= e^{-i\lambda \int_{\mathbb{R}^4} \frac{\delta^4}{\delta J^4}} e^{\frac{i}{2} \int_{x,z_1} J(x) G(x,z_1) J(z_1)} \left[G(x,y) + \int_{z_1, z_2} G(x, z_1) G(y, z_2) J(z_1) J(z_2) \right] \end{aligned}$$

1st order in λ

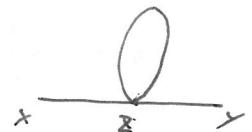
$$\approx \left(1 - i\lambda \int_{\mathbb{R}^4} \frac{\delta^4}{\delta J^4} \right) \left(1 + \frac{1}{2} \int_{x,z_1} J(x) G(x, z_1) J(z_1) + \frac{1}{2!} \left(\frac{1}{2}\right)^2 \int_{x,z_1} \int_{y,z_2} J(x) G(x, z_1) J(z_1) G(y, z_2) J(y) + \dots \right) \left[G(x,y) + \int_{z_1, z_2} \dots \right] \Big|_{J=0}$$

$$= G(x,y)$$

results.

$$-i\lambda \int_{\mathbb{R}^4} \frac{\delta^4}{\delta J^4} \frac{1}{2} \int_{x,z_1} J(x) G(x, z_1) J(z_1) \int_{y,z_2} G(y, z_2) J(y) \Rightarrow -i\lambda \frac{1}{2} \int_{\mathbb{R}^4} G(x, z_1) G(y, z_2) G(z_1, z_2)$$

$$-i\lambda G(x,y) \int_{\mathbb{R}^4} \frac{\delta^4}{\delta J^4} \frac{1}{8} \int_{x,z_1} \int_{y,z_2} J(x) G(x, z_1) J(z_1) \int_{z_3, z_4} G(y, z_2) G(z_2, z_3) G(z_3, z_4)$$



negligible in physics

$$x \longrightarrow y \quad B_z$$

(disconnected)

vacuum correction.

$$\tilde{G}(x,y) \approx G(x,y) - i\lambda \underbrace{\frac{1}{2} \int_{\mathbb{R}^4} G(x, z_1) G(y, z_2) G(z_1, z_2)}_{\text{mass correction to the particle.}}$$

$$= \int_{(2i)^4} \frac{d^4 k}{(2i)^4} e^{ik(x-y)} \left[\frac{i}{k^2 - m^2 + i\epsilon} + \frac{i}{k^2 - m^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon} + \dots \right]$$

skip
補

$$\frac{i}{k^2 - (m^2 + \delta m^2) + i\epsilon} = \frac{i}{k^2 - m^2 + i\epsilon} + \frac{i}{k^2 - m^2 + i\epsilon} \frac{\delta m^2}{k^2 - m^2 + i\epsilon}$$

$$\delta m^2 = 12\lambda G(z, z) + O(\lambda^2)$$

$$G(z, z) \approx G(0) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \quad P_0 = i P_4$$

$$= \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{P_E^2 + m^2} \quad P_E^2 = P_4^2 + \vec{p}^2$$

$$= \frac{\pi^2}{16\pi^4} \int_0^\Lambda \frac{p^2 dp^2}{p^2 + m^2} \quad d^4 p_E \rightarrow 2\pi^2 p^3 dp \\ = \pi^2 p^2 dp^2$$

$$= \frac{1}{16\pi^2} \int_0^{\Lambda^2} \frac{y dy}{y + m^2}$$

$$\overbrace{\int_0^{\Lambda^2} \left(1 - \frac{m^2}{y + m^2}\right) dy} = \Lambda^2 - m^2 \log \frac{\Lambda^2}{m^2}$$

$$\Rightarrow \delta m^2 = \frac{3\lambda}{4\pi^2} \left(\Lambda^2 - m^2 \log \frac{\Lambda^2}{m^2} \right) \quad \Lambda : \text{cut-off.}$$

difference is observable. (changing the environment)

$$\left. \delta(m^2) \right|_{\text{Environment}} - \left. \delta(m^2) \right|_{\text{vacuum}} = \text{observable} \quad \underline{\text{interaction effect.}}$$

Simplify $Z[J] = e^{i\lambda \int_x \frac{\delta^4}{8J^4}} e^{\frac{i}{2} \int_{x,y} J(x) G(x,y) J(y)} = F[\frac{\delta}{\delta J}] Z_0[J]$

Using

Identity

$$F[\frac{\delta}{\delta J}] H[J] = H[\frac{\delta}{\delta \varphi}] F[\varphi] e^{\int J \varphi} \Big|_{\varphi=0}$$

Proof

$$\begin{aligned} & H[\frac{\delta}{\delta \varphi}] F[\varphi] e^{\int J \varphi} \Big|_{\varphi=0} && \varphi : \text{arbitrary function.} \\ & \underbrace{F[\frac{\delta}{\delta J}] e^{\int J \varphi}}_{F[\frac{\delta}{\delta J}] H[J] e^{\int J \varphi}} \Big|_{\varphi=0} && = F[\frac{\delta}{\delta J}] H[J]. \quad \checkmark \end{aligned}$$

$$\Rightarrow Z[J] = Z_0[\frac{\delta}{\delta \varphi}] F[\varphi] e^{\int J \varphi} \Big|_{\varphi=0}$$

$$= e^{\frac{i}{2} \int \frac{\delta}{\delta \varphi} G \frac{\delta}{\delta \varphi}} e^{\int J \varphi} e^{-i\lambda \int \varphi^4} \underbrace{e^{\int J \varphi} e^{-\int J \varphi} e^{\frac{i}{2} \int G \varphi \varphi} e^{\int J \varphi}}_{e^{\frac{i}{2} \int G \varphi \varphi}}$$

$$e^{\frac{i}{2} \int G \varphi \varphi} = [\int J \varphi, \frac{i}{2} \int G \varphi \varphi] + \frac{1}{2!} [\int J \varphi, [\int J \varphi, \frac{i}{2} \int G \varphi \varphi]] + \dots$$

$$e^{-A} B e^A = B - [A, B]$$

$$+ \frac{1}{2!} [A, [A, B]] + \dots$$

$$e^{-A} e^B e^A \Rightarrow e^{(e^{-A} B e^A)}$$

$$Z[J] = \underbrace{e^{\frac{i}{2} \int Q \delta \delta}}_{\delta = 0} e^{\int J \varphi} e^{-i \lambda \int \varphi^4}$$

$$\int Q \delta \delta = \int_{x,y} Q(x,y) \frac{\delta}{\delta g(x)} \frac{\delta}{\delta g(y)}$$

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$$e^{\int J \varphi} \underbrace{e^{-\int J \varphi} e^{\frac{i}{2} \int Q \delta \delta} e^{\int J \varphi}}_{\exp \left[e^{-\int J \varphi} \underbrace{\frac{i}{2} \int Q \delta \delta}_{\delta = 0} e^{\int J \varphi} \right]}$$

$$e^{-A} e^B e^A = e^{e^{-A} B e^A}$$

$$e^{-A} B e^A = B + [B, A] + \frac{1}{2!} [[B, A], A] + \dots$$

$$\frac{i}{2} \int Q \delta \delta + \underbrace{[\frac{i}{2} \int Q \delta \delta, \int J \varphi]}_{\delta = 0} + \frac{1}{2!} [[\frac{i}{2} \int Q \delta \delta, \int J \varphi], \int J \varphi] + \dots$$

$$\int \frac{i}{2} Q \delta [\delta, \int J \varphi] + \int \frac{i}{2} Q [\delta, \int J \varphi] \delta = \int Q J \delta$$

$$= \int_{x,y} J(x) Q(x,y) \frac{\delta}{\delta g(x)}$$

$$[\int Q J \delta, \int J \varphi] = \int Q J [\delta, \int J \varphi] = \int_{x,y} J(x) Q(x,y) J(y)$$

$$\rightarrow Z[J] = e^{\int J \varphi} e^{\frac{i}{2} \int Q \delta \delta} e^{\int J Q \delta} e^{\frac{i}{2} \int J Q J} e^{-i \lambda \int \varphi^4} \Big|_{\varphi = 0}$$

$$= e^{\frac{i}{2} \int J Q J} \left[e^{\int J Q \delta} \mathcal{F}[\varphi] \right]_{\varphi = 0}$$

$$\mathcal{F}[\varphi] = e^{\frac{i}{2} \int Q \delta \delta} e^{-i \lambda \int \varphi^4}$$

$$\begin{aligned}
 Z[J] &= \langle 0 | T e^{\int J\phi} | 0 \rangle \\
 &= e^{-i\lambda \int \frac{\delta^4}{\delta \phi^4}} e^{\frac{i}{2} \int J G J} \\
 &= e^{\frac{i}{2} \int J G J} e^{\int J G \frac{\delta}{\delta \phi}} e^{\frac{i}{2} \int G \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi}} e^{-i\lambda \int \phi^4} \Big|_{\phi=0} \\
 &= e^{\frac{i}{2} \int J G J} e^{\int J G \frac{\delta}{\delta \phi}} \mathcal{F}[\phi] \Big|_{\phi=0}
 \end{aligned}$$

In general

$$\mathcal{F}_i[\phi] = e^{\frac{i}{2} \int G \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi}} e^{i S_{\text{int}}[\phi]} \quad \text{S-matrix functional}$$

Wick contraction operator

replace a pair of ϕ by G

[quantum effect] $\circlearrowleft G$

classical.

$\frac{i}{\hbar} S_{\text{int}}$

As far as the scattering amplitude is concerned

$$e^{\frac{i}{2} \int J G J} \Rightarrow 1$$

$$e^{\int J G \frac{\delta}{\delta \phi}}$$

↑

replace ϕ by JG

- No perturbation approx.
- any polynomial type interaction
- Derived from the Heisenberg eqn. of motion & the equal-time CR's.

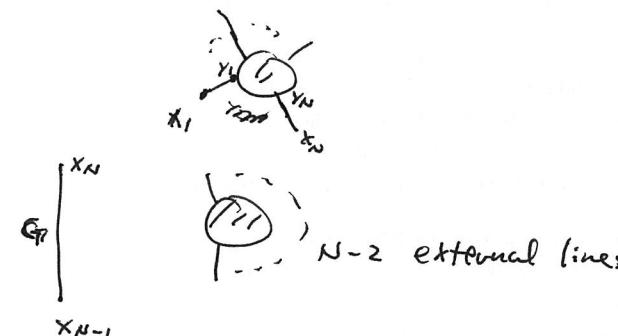
$$\Sigma[J] = e^{\frac{i}{2} \int J G J} e^{\int J \frac{\delta}{\delta \phi}} \mathcal{F}[\phi] \Big|_{\phi=0}$$

N-point function

$$\frac{\delta^N}{\delta J^N} \Sigma[J] \Big|_{J=0}$$

$$\begin{array}{c} \overbrace{\hspace{1cm}} \\ 0 \end{array} \quad \begin{array}{c} \overbrace{\hspace{1cm}} \\ N \end{array}$$

$N-2$



Scattering amplitudes

$$e^{i \int J G J} = 1$$

fly-by (do not participate in the process)

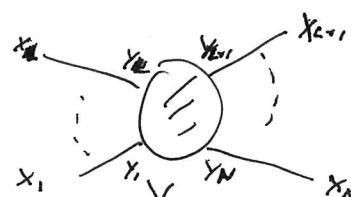
$$e^{\int J \frac{\delta}{\delta \phi}} \mathcal{F}[\phi] = \sum_N \frac{1}{N!} \left(\int_{x_1} J G \dots \int_{x_N} J G \right) V(x_1 \dots x_N)$$

↑
Replace ϕ
by JG

Assuming $\mathcal{F}[\phi] = \frac{1}{N!} \sum_N \int_{x'_s} \phi(x_1) \dots \phi(x_N) \underline{V}(x_1 \dots x_N)$

N-point correlator

$$G_N(x_1 \dots x_N) = \int_{x'_s} G(y_1, x_1) G(y_2, x_2) \dots G(y_N, x_N) \underline{V}(x_1 \dots x_N)$$



This carries
interaction information

LSD reduction formula.

Let $x_1^\circ \dots x_M^\circ \rightarrow -\infty$
 $x_{M+1}^\circ \dots x_N^\circ \rightarrow +\infty$

$$G(x, y) = \begin{cases} \sum_k u_k(x) u_k^*(y) & x^\circ > y^\circ \\ \sum_k u_k^*(x) u_k(y) & y^\circ > x^\circ \end{cases}$$

$G_N = \langle x_{M+1} \dots x_N | x_1 \dots x_M \rangle$ scattering amp. in position-diagonal rep.

$$G_N(x_1 \dots x_N) = \int d^k z_1 \dots d^k z_N \sum_{\substack{k_i \\ p's}} u_{k_1}(z_1) u_{k_1}^*(x_1) \dots u_{k_M}(z_M) u_{k_M}^*(x_M) \\ u_{p_{M+1}}(x_{M+1}) u_{p_{M+1}}^*(z_{M+1}) \dots u_{p_N}(x_N) u_{p_N}^*(x_N) V(x_1 \dots x_N)$$

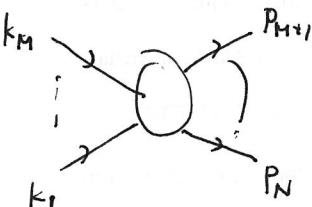
$$V(x_1 \dots x_N) = \frac{\delta^N}{\delta \varphi^N} \mathcal{F}[\varphi] \Big|_{\varphi=0}$$

$$\sum_k u_{k_1}^*(x_1) \dots u_{k_M}^*(x_M) = 1.$$

$$\sum_{k_i} u_{k_i}^*(x_i) = \int \frac{d^3 k}{(2\pi)^3} 2\omega$$

In the momentum-space rep.

$$\langle p_{M+1} \dots p_N | k_1 \dots k_M \rangle = \langle k_1 \dots k_M \rightarrow p_{M+1} \dots p_N \rangle = \int_{z_1 \dots z_N} u_{k_1}(z_1) \dots u_{k_M}(z_M) u_{p_{M+1}}^*(z_{M+1}) \dots u_{p_N}^*(z_N)$$



$$x \frac{d}{d\varphi(z_1)} \dots \frac{d}{d\varphi(z_N)} \mathcal{F}[\varphi] \Big|_{\varphi=0}$$

Scattering amplitudes

$$\langle p_N | x_N \rangle \dots \langle p_{M+1} | x_{M+1} \rangle \underbrace{\langle x_{M+2} \dots x_N | x_1 \dots x_M \rangle}_{\otimes u_{k_M}^*} \langle x_{M+1} | k_M \rangle \dots \langle x_1 | k_1 \rangle$$

$$= \langle p_{M+1} \dots p_N | k_1 \dots k_M \rangle$$

$\otimes u_{k_M}^*$

momentum space HS

$$\sum_{k_M} u_{k_M} u_{k_M}^*$$

Recapitulate

$$\xi = S_0 + S_{\text{int}}$$

$$\Sigma = e^{\frac{i}{2} \int J G J} \left[e^{\int J G \delta} \mathcal{F}[\varphi] \right]_{\varphi=0}$$

$$\mathcal{F}[\varphi] = e^{\frac{i}{2} \int G \delta \varphi} e^{i S_{\text{int}}}$$

inner product $(u_k | u_\ell) = \delta_{k\ell}$.
 $u_k | u_\ell$ 遍立.

$$\delta_{k,k'} = \frac{(2\pi)^3}{V} \delta^{(3)}(k - k')$$

$$(u | v) = \int d^3x (u^\dagger i \partial_\mu v - i \partial_\mu u^\dagger v)$$

$$(u_k | u_\ell) = \delta_{k\ell}$$

$$\mathcal{A}(k_1 \dots k_M \rightarrow p_{M+1} \dots p_N) = \int d^4 z_1 \dots d^4 z_N u_{k_1}(z_1) \dots u_{p_{M+1}}^*(z_{M+1}) \dots \frac{\delta^N}{\delta \varphi(x_1) \dots \delta \varphi(x_N)} \mathcal{F}[\varphi] \Big|_{\varphi=0}$$

Integrations over $d^4 z$'s \rightarrow four-momentum conservation



$$\mathcal{A}(k \rightarrow p) = \left(\prod_{i=1}^N \frac{1}{\sqrt{2\omega_i} V} \right) (2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^M k_i - \sum_{f=M+1}^N p_f \right) M$$

M matrix element.
calculated from the above

Decay rates & cross section

ow to calculate decays & scattering

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(1) Decay rate Γ

$$A = \frac{1}{\sqrt{2\omega_k V}} \sum_{i=1}^n \frac{1}{\sqrt{2\omega_i V}}$$

Decay probability

$$|\mathcal{A}|^2 = \frac{1}{2\omega_{KV}} \cdot \frac{1}{2\omega_{IV}} \cdot (2\pi)^4 \int^{(4)}_{} \left(k - \sum_{i=1}^n p_i \right) VT |$$

$$\Gamma = \frac{|\mathcal{A}|^2}{T} = \frac{1}{2\omega_k} \prod_i \frac{1}{2\omega_i v} (2\pi)^4 \delta^{(4)}(k - \sum_{i=1}^n p_i) |\mathcal{M}|^2$$

In practice, we take a sum over all outgoing state

$$d\Gamma = \frac{1}{2\omega_k} |M|^2 (z_{ii})^4 f(k - \sum_{i=1}^n p_i)$$

Total decay rate Not Lorentz inv.

$$\Gamma_{\text{tot}} = \int d\Gamma$$

$$\tau = \frac{1}{P_{\text{tot}}} \quad \text{life time.}$$

$$\begin{aligned}
 & \underline{(2\pi)^4 \int^{(4)} \left(k - \sum_{i=1}^n p_i \right) M } \\
 & \quad \left(\int dx e^{i(k-\sum p)x} \int dy e^{i(k-\sum p)y} \right) \\
 & = (2\pi)^4 \int^{(4)} (k - \sum p) \int dy \underline{VT} |M|^2
 \end{aligned}$$

range of time integration

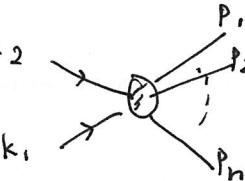
The diagram illustrates the decomposition of a vector E into components d^3P_1 and d^3P_2 . The vector E is shown originating from a point and pointing towards the upper right. Two perpendicular axes, represented by dashed lines, define the planes for the decomposition. The component d^3P_1 is shown as a vector along the upper axis, and the component d^3P_2 is shown as a vector along the lower axis, both originating from the same point as E .

$$\prod_i \frac{\partial P_i}{2\omega_i (2\pi)^3} \text{ Lorentz invariant} \left(\frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_p} = \frac{d^4 p}{(2\pi)^4} 2\pi f(p^2 - m^2) \Theta(p^0) \right)$$

V, T cancel out

$\omega_k \rightarrow$ time-dilation effect for τ 's.
for fast moving particles

④ Cross sections.



$$S = \frac{1}{\sqrt{2\omega_{k_1}V}} \frac{1}{\sqrt{2\omega_{k_2}V}} \prod_i^n \frac{1}{\sqrt{2\omega_iV}} (2\pi)^4 \delta^{(4)}(\sum k - \sum p) M.$$

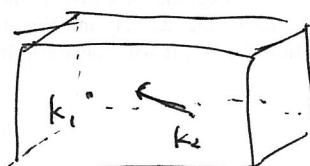
Transition rate

$$\frac{|\mathcal{A}|^2}{T} = \frac{1}{2\omega_{k_1}} \frac{1}{2\omega_{k_2}V} \prod_i \frac{1}{2\omega_iV} (2\pi)^4 \delta^{(4)}(\sum k - \sum p) |M|^2$$

Cross section

$$d\sigma = \frac{(\text{Transition rate})}{(\text{Flux})}$$

k_1 -rest frame



$$\vec{F} = \frac{1}{V} (\text{velocity of } k_2)$$

$$= \frac{1}{V} \frac{|\vec{k}_2|}{\omega_{k_2}}$$

In general

$$F = \frac{1}{V} \frac{\sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2}}{\omega_{k_1} \omega_{k_2}}$$

k_2 -rest frame

$$F = \frac{1}{V} \frac{|\vec{k}_1|}{\omega_{k_1}}$$

$$\omega_k = \sqrt{E + m^2}$$

$$\omega_k = \frac{m}{\sqrt{1-v^2}}$$

$$\vec{k} = \frac{m\vec{v}}{\sqrt{1-v^2}}$$

Symmetric in $1 \leftrightarrow 2$.

$$d\sigma = |M|^2 \frac{(2\pi)^4 \delta^{(4)}(\sum k - \sum p)}{4\sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2}} \prod_i \frac{d^3 p_i}{2\omega_{p_i} (2\pi)^3}$$

Lorentz invariant

VT factor cancels out

5. Electromagnetic field.

$$\delta \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

$$= -\frac{1}{2} F_{\mu\nu} \delta F^{\mu\nu}$$

$$+ (\partial_\mu A^\nu - \partial_\nu A^\mu) = \frac{1}{2} (E^2 - B^2) - A_\mu J^\mu$$

$$= -F_{\mu\nu} \partial^\mu A^\nu$$

$$\delta S = 0 \Rightarrow \partial_\mu F^{\mu\nu} = J^\nu$$

Quantization.

Difficulties



canonical 1-form

(E_i, A_i)

$\begin{matrix} \uparrow \\ (x_i, p_i), (\phi, \dot{\phi}) \end{matrix}$

1. Redundancy in

$$A \rightarrow A'_\mu = A_\mu + \partial_\mu f.$$

$$F'_{\mu\nu} = F_{\mu\nu} \quad \text{fix: infinity of freedom}$$

2. The Gauss law $\nabla \cdot \vec{E} = J^0$

can not be derived as the Heisenberg eqn of motion.

appropriate for Θ .

Choose

$$A_0 = 0$$

$$A_i = A_i^T + \partial_i f$$

$$\text{with } \partial_0 A_i^T = 0 \quad (\text{Transverse, round.})$$

determine by
Gauge law (as a constraint)

(E_i, A_i) form

a set of phase space variables.

$$\rho \vec{J}$$

$$A_\mu = (A_0, A_i) \quad J^\mu = (J^0, J^i)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{0i} = E_i$$

$$F_{ij} = \epsilon_{ijk} B_k$$

$$E_i = \partial_0 A_i - \partial_i A_0$$

$$B_i = \epsilon_{ijk} \partial_j A_k$$

Maxwell eqn

$$\nu = 0$$

$$\partial_\mu F^{\mu 0} = J^0 \rightarrow$$

$$\partial_i E^i = J^0$$

$$\nu = j$$

$$\partial_\mu F^{\mu j} = J^j \rightarrow$$

$$\partial_0 E^j + (\nabla \times B)^j = J^j$$

$$\partial_0 E^j - \partial_i (\epsilon^{ijk} B_k)$$

$$- (\nabla \times B)^j$$

identity (check)

$$\nabla \cdot \vec{B} = 0$$

$$-(\nabla \times \vec{E}) + \partial_0 \vec{B} = 0$$

$$F_{\mu\nu} F^{\mu\nu} = \underbrace{F_{0V} F^{0V}}_{-F_{0i} F^{0i}} - \underbrace{F_{iV} F^{iV}}_{F_{i0} F^{i0} - F_{ij} F^{ij}} = -2E^2 - 2B^2$$

second order time derivative

$$\text{OK for } \partial_0 \vec{E} + (\nabla \times \vec{B}) = \vec{J}.$$

$$\partial_0 A_i - \partial_i A_0$$

at fixed time

Then,

$$\begin{aligned} A_{\mu} &= (0, A_i) \\ &\stackrel{\text{"}}{=} A_i^T + \partial_i f \\ &\quad \nabla A^T = 0 \end{aligned}$$

$$F_i = \partial_0 A_i = \partial_0 (A_i^T + \partial_i f) = \partial_0 A_i^T + \partial_i (\partial_0 f)$$

$$B_i = \epsilon_{ijk} \partial_j A_k^T$$

Maxwell eqns.

$$\partial_i F_i = J_0$$

$$\rightarrow \boxed{\nabla^2(\partial_0 f) = J_0} \quad f \text{ is determined by } J_0. \quad f[J_0]$$

$$\partial_0 F_i + (\nabla \times B)_i = J_i$$

Poisson eqn

$$\boxed{\partial_0 f(x^0, \vec{x}) = -\frac{1}{4\pi} \int d^3y \frac{1}{|\vec{x}-\vec{y}|} J_0(x^0, \vec{y})} = \int d^3y G_c(\vec{x}, \vec{y}) J_0(x^0, \vec{y})$$

$$\nabla^2 G_c(\vec{x}, \vec{y}) = +\delta^{(3)}(\vec{x} - \vec{y})$$

$$\boxed{G_c(\vec{x}, \vec{y}) = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot (\vec{x}-\vec{y})}}{\vec{k}^2} = \frac{1}{4\pi} \frac{1}{|\vec{x}-\vec{y}|}}$$

$$\begin{aligned} \nabla^2(\partial_0 f) &= \int d^3y \delta^{(3)}(\vec{x}-\vec{y}) J_0(x^0, \vec{y}) \\ &= J_0(x^0, \vec{x}) \end{aligned}$$

Coulomb Green's function.

④ The remaining A_i^T should be quantized.

$$\begin{aligned} \frac{1}{2} \int E^2 &= \frac{1}{2} \int \partial_0 A_i^T \partial_0 A_i^T + \int \underbrace{\partial_0 A_i^T \partial_i (\partial_0 f)}_{0} + \underbrace{\frac{1}{2} \int \partial_i (\partial_0 f) \partial_i (\partial_0 f)}_{J_0} \\ &\quad \partial_i (\partial_0 A_i^T \partial_0 f) - \cancel{\partial_0 \partial_i A_i^T (\partial_0 f)} - (\partial_0 f) \nabla^2(\partial_0 f) \end{aligned}$$

$$\frac{1}{2} \int E^2 = \frac{1}{2} \int \partial_\mu A_j^\tau \partial_\mu A_j^\tau - \frac{1}{2} \int (\partial_\mu f) J_0$$

$$\begin{aligned} \frac{1}{2} \int B^2 &= \frac{1}{2} \int (\epsilon_{ijk} \partial_j A_k^\tau - \epsilon_{ilm} \partial_l A_m^\tau) \\ &= \frac{1}{2} \int \underbrace{(\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) \partial_j A_k^\tau \partial_l A_m^\tau}_{\partial_j A_k^\tau \partial_j A_k^\tau - \underbrace{\partial_j A_k^\tau \partial_k A_j^\tau}_{\partial_j (A_k^\tau \partial_k A_j^\tau)} - \underbrace{A_k^\tau \partial_k \partial_j A_j^\tau}_{0}} = \frac{1}{2} \int \partial_i A_j^\tau \partial_i A_j^\tau \\ - \int A_\mu J^\mu &= \int (A_i^\tau + \partial_i f) J_i \\ &= \int A_i^\tau J_i + \underbrace{\int \partial_i f J_i}_{-\cancel{f} \underbrace{\partial_i J_i}_{\partial_\mu J^\mu = 0} + \cancel{f} \partial_\mu J^\mu} = \int A_i^\tau J_i + \cancel{(2f) J_0} \quad \text{Conservation of current} \end{aligned}$$

$$\partial_\mu J^\mu = 0 \Rightarrow \partial_\mu J_0 = \partial_i J_i$$

$$\underbrace{- \int f \partial_\mu J_0}_{\sim \int f \partial_\mu J_0} = \int (\partial_\mu f) J_0$$

$$S = \int \frac{1}{2} (E^2 - B^2) - \int A \cdot J$$

$$\begin{aligned} &\approx \int \frac{1}{2} (\partial_\mu A_j^\tau \partial_\mu A_j^\tau - \partial_i A_j^\tau \partial_i A_j^\tau) + \int A_i^\tau J_i + \frac{1}{2} \underbrace{\int (\partial_\mu f) J_0}_{\int d^4x d^3y G_c(x, \vec{y}) J_0(x, \vec{y}) J_0(x, \vec{x})} \end{aligned}$$

$$S = \frac{1}{2} \int (\partial_\mu A_i^\dagger \partial_\mu A_i^\dagger) + \int A_i^\dagger J_i + \frac{1}{2} \int d^4x d^4y J_0(x^0, \vec{x}) G_c(\vec{x} - \vec{y}) \delta(x^0 - y^0) J_0(y^0, \vec{y})$$

$\underbrace{\quad\quad\quad}_{\partial_\mu A_i^\dagger = 0}$ $\underbrace{\quad\quad\quad}_{\text{free theory}}$ $\underbrace{\quad\quad\quad}_{\text{electrostatic interaction.}}$
 Two polarization states for photon absorption & emission of photons
 Eqn of motion $\square A_i^\dagger = 0$ Lorentz force effect.

mode expansion



$$A_i^\dagger(x) = \sum_{k,\lambda} \left[a_{k,\lambda} e_i^{(k)} u_k(x) + a_{k,\lambda}^* e_i^{(k)*} u_k^*(x) \right]$$

$$u_k(x) = \frac{e^{-ik \cdot x}}{\sqrt{2\omega_k V}} \quad \omega_k = \sqrt{k^2}$$

$$\partial_\mu A_i^\dagger = 0$$

polarization $e_i^{(k)}$
vector



$$e_i^{(k)} \cdot k_i = 0$$

$$\sum_j e_i^{(k)} e_j^{(k)} = \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \quad \begin{bmatrix} \text{symmetric in } i \leftrightarrow j \\ k_i()_{;j} = k_j()_{;i} = 0 \end{bmatrix}$$

Commutation relations

$$[a_{k,\lambda}, a_{k',\lambda'}] = 0$$

$$[a_{k,\lambda}^*, a_{k',\lambda'}^*] = 0$$

$$[a_{k,\lambda}, a_{k',\lambda'}^*] = \delta_{kk'} \delta_{\lambda\lambda'}$$

$$k^2 = k_0^2 - \vec{k}^2$$

$(\omega_k = k_0)$

Same as the scalar field except the $e_i^{(k)}$ contribution

④ Propagator

$$D_{ij}(x, y) = \langle 0 | T A_i^\dagger(x) A_j^\dagger(y) | 0 \rangle$$

$$= \int \frac{d^4 k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 + i\epsilon} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \quad \text{Photon propagator}$$

⑤ S-matrix functional

$$\begin{aligned} \mathcal{F}_1[A] &= e^{\frac{1}{2} \int d^4x d^4y D_{ij}(x, y) \frac{\delta}{\delta A_i^\dagger(x)} \frac{\delta}{\delta A_j^\dagger(y)}} e^{i S_{\text{int}}} \\ &= e^{\frac{1}{2} \int D_{ij} \frac{\delta}{\delta A_i^\dagger} \frac{\delta}{\delta A_j^\dagger}} e^{i \frac{1}{2} \int J_0 G_c J_0 + i \int A_i^\dagger J_i} \end{aligned}$$

quadratic in J

$$\begin{aligned} \mathcal{F}^{(2)} &= \frac{i}{2} \int_{x,y} J_0 G_c J_0 - \underbrace{\frac{1}{2} \int_{x,y} J_i D_{ij} J_j}_{\int_{x,y} J_i(x) J_j(x) \int_k \frac{i e^{-ik(x-y)}}{k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right)} \\ &= \int_{xy} J_i(x) J_j(y) \int \frac{i e^{-ik(x-y)}}{k^2} + \int_{xy} \cancel{J_i(x) J_j(y)} \partial_{x_i} \partial_{y_j} \cancel{\left[\frac{i e^{-ik(x-y)}}{k^2 + i\epsilon} \right]} \end{aligned}$$

$$\mathcal{F}_1^{(2)} = \underbrace{i \frac{1}{2} \int_{x,y} J_0(x) \underbrace{G_\epsilon(\vec{x}-\vec{y})}_{-\int \frac{d^3k}{(2\pi)^3} \frac{e^{ik \cdot (\vec{x}-\vec{y})}}{k^2}} \delta(x^0-y^0) J_0(y)}_{-\int \frac{dk_0}{2\pi} e^{-ik_0(x^0-y^0)}} - \frac{1}{2} \int_{x,y} J_i(x) D_{ij}(x,y) J_j(y)$$

$$\int_{x,y,k} J_i(x) J_j(y) \frac{i e^{-ik \cdot (x-y)}}{k^2 + i\epsilon} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right)$$

$$- \frac{i}{2} \int_{x,y,k} J_0(x) \frac{e^{-ik \cdot (x-y)}}{k^2} J_0(y)$$

$$= \int J_i(x) J_i(y) \frac{i e^{-ik \cdot (x-y)}}{k^2 + i\epsilon}$$

$$= - \frac{i}{2} \int_{x,y} J_0(x) J_0(y) \int_k \frac{e^{-ik \cdot (x-y)}}{k^2} \\ + \frac{i}{2} \int_{x,y} J_0(x) J_0(y) \int_k \frac{k_0^2}{k^2} \frac{i e^{-ik \cdot (x-y)}}{k^2 + i\epsilon} \\ - \frac{1}{2} \int_{x,y} J_i(x) J_i(y) \int_k \frac{i e^{-ik \cdot (x-y)}}{k^2 + i\epsilon}$$

$$- \int J_i(x) J_j(y) \underbrace{\frac{\partial x_i \partial y_j}{k^2}}_{\begin{array}{l} \text{---} \\ \text{---} \end{array}} \left[\frac{i e^{-ik \cdot (x-y)}}{k^2 + i\epsilon} \right]$$

$$\int_{x,y,k} \underbrace{\partial_i J_i}_{\partial_0 J_0} \underbrace{\partial_j J_j}_{\partial_0 J_0} \frac{1}{k^2} \left[\begin{array}{l} \text{---} \\ \text{---} \end{array} \right]$$

$$= \int J_0(x) J_0(y) \frac{k_0^2}{k^2} \left[\begin{array}{l} \text{---} \\ \text{---} \end{array} \right]$$

$$\frac{1}{2} \int_{x,y} J_0 J_0 \int_k \frac{i e^{-ik \cdot (x-y)}}{k^2 + i\epsilon} \frac{1}{k^2} \left[k_0^2 - \underbrace{(k^2 + i\epsilon)}_{k_0^2 - k^2} \right] \\ = 1$$

$$\mathcal{F}_1^{(2)} = \frac{1}{2} \int_{xy} J_\mu^\nu(x) J_\nu^\mu(y) D_{\mu\nu}(x,y)$$

Lorentz covariant form.

$$D_{\mu\nu}(x,y) = \eta_{\mu\nu} \int \frac{d^3k}{(2\pi)^3} \frac{i e^{-ik \cdot (x-y)}}{k^2 + i\epsilon}$$

Γ
 $(1_{-1,-1})$

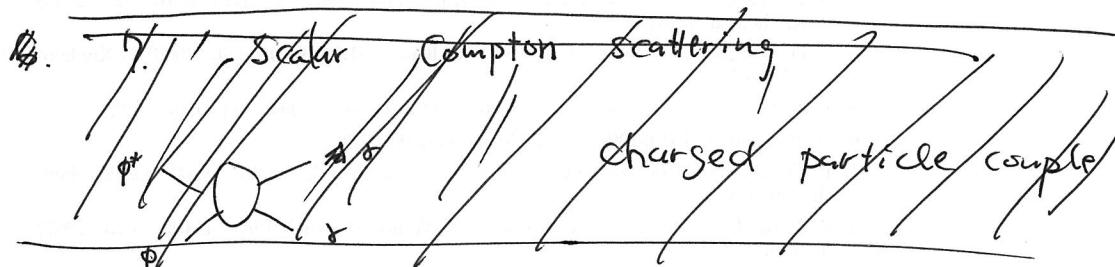
The propagator, when applied to conserved currents,
can be taken as the covariant propagator $D_{\mu\nu}(x, y)$!

$$\mathcal{J}^{(2)} = -\frac{1}{2} \int_{x,y} D_{\mu\nu}(x, y) \frac{\delta}{\delta A_\mu} \frac{\delta}{\delta A_\nu} \frac{(i)^2}{2!} \int_A A_\mu J^\mu \int A_\nu \bar{A}^\nu$$

$$\Rightarrow \boxed{\mathcal{F}[A] = e^{-\frac{1}{2} \int_{x,y} D_{\mu\nu}(x, y) \frac{\delta}{\delta A_\mu} \frac{\delta}{\delta A_\nu} e^{i S_{\text{int}}}}}$$

~~apply to any order~~
~~in/any/a~~

↑
Wick contraction operator.



The only renormalizable
& unitary coupling one

Interaction with charged particles

↑
complex scalar fields

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi \phi^*$$

invariant under

$$\phi \rightarrow e^{i Q \theta} \phi$$

Gauge principle

$$\partial_\mu \rightarrow D_\mu = \partial_\mu - i Q A_\mu$$

$$\Rightarrow \mathcal{L} = (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi \phi^*$$

$$= \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^* + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_{\text{int}} = -iQ A_\mu (\phi \partial^\mu \phi^* - \partial^\mu \phi \phi^*) + Q^2 A_\mu A^\mu \phi \phi^*$$

$$\mathcal{F}[\phi, A] = e^{\int G \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi^*}} e^{-\frac{i}{2} \int P_{\mu\nu} \frac{\delta}{\delta A_\mu} \frac{\delta}{\delta A^\nu}} e^{i S_{\text{int}}}$$

6. List of S-matrix functionals.

Scalar field

$$\mathcal{F}[\phi] = e^{\frac{i}{2} \int G \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi}} e^{i S_{\text{int}}[\phi]}$$

charged scalar. π^\pm, k^\pm
(pseudo)scalar

Complex scalar field

$$\mathcal{F}[\phi, \phi^*] = e^{\int G \frac{\delta}{\delta \phi} \frac{\delta}{\delta \phi^*}} e^{i S_{\text{int}}[\phi, \phi^*]}$$

Weak EM field

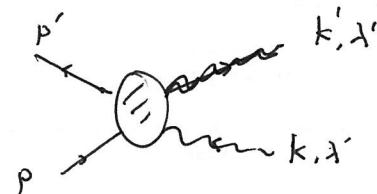
$$\mathcal{F}[A] = e^{-\frac{i}{2} \int P_{\mu\nu} \frac{\delta}{\delta A_\mu} \frac{\delta}{\delta A^\nu}} e^{i \cancel{S}_{\text{int}} + \int A_\mu J^\mu}$$

Dirac field

$$\mathcal{F}[\bar{\psi}, \bar{\psi}] = e^{-\int_{x,y} S_{\text{int}}(x, y) \frac{\delta}{\delta \bar{\psi}_x(x)} \frac{\delta}{\delta \bar{\psi}_y(y)}} e^{i S_{\text{int}}}$$

QED

$$\mathcal{F}[\bar{\psi}, \bar{\psi}, A] = e^{-\frac{i}{2} \int P_{\mu\nu} \frac{\delta}{\delta A_\mu} \frac{\delta}{\delta A^\nu}} e^{-\int S(x, y) \frac{\delta}{\delta \bar{\psi}_x(x)} \frac{\delta}{\delta \bar{\psi}_y(y)}} e^{i \int A_\mu \bar{\psi} \gamma^\mu \psi}$$



scalar Compton scattering

$$S(x, y) = \int \frac{dk}{(2\pi)^d} \frac{i e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} + (j \cdot k + m)$$

Fermion propagator

$$S_{\text{int}}(x, y) = \langle \delta(T \bar{\psi}(x) \bar{\psi}(y)) \rangle$$

$$= i \int \frac{dk}{(2\pi)^d} \frac{(j \cdot k + m)}{k^2 - m^2 + i\epsilon} e^{-ik(x-y)}$$

$$= (i j \cdot k + m) G(x, y)$$

S^a -matrix